Word automaticity of tree automatic ordinals is decidable

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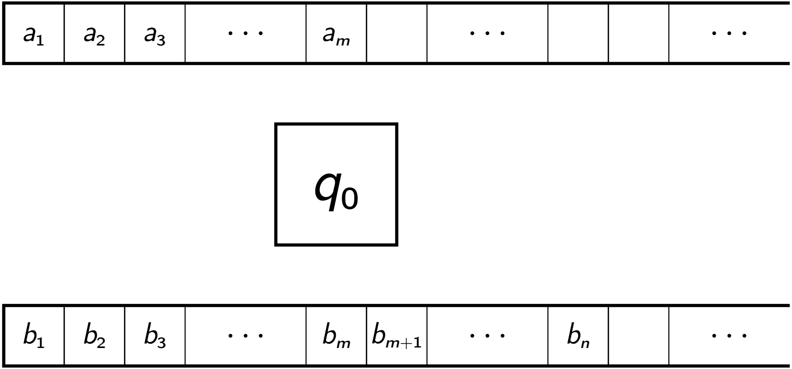
Introduction

- In the 1960s, Büchi, Elgot, Rabin, and Trakhtenbrot used finite automata on (infinite) words and trees to obtain positive decidability results for several logical theories.
- In 1995, Khoussainov and Nerode initiated the study of **automaton presentable** structures, i.e., structures presented by finite automata, and showed that every such structure has decidable first order theory.
- Later on, these investigations were extended to structures presented by automata on infinite words and (infinite) trees.
- Since trees generalize words, every structure presentable by finite automata is also presentable by tree automata.
- However, there are structures presesentable by tree automata which are not presentable by finite automata.
- Naturally, the following question arises: Given a structure presesented by tree automata, is it decidable, whether this structure is also presentable by finite automata?

Synchronous multi-tape automata

Question

How can we present relations on words by finite automata?



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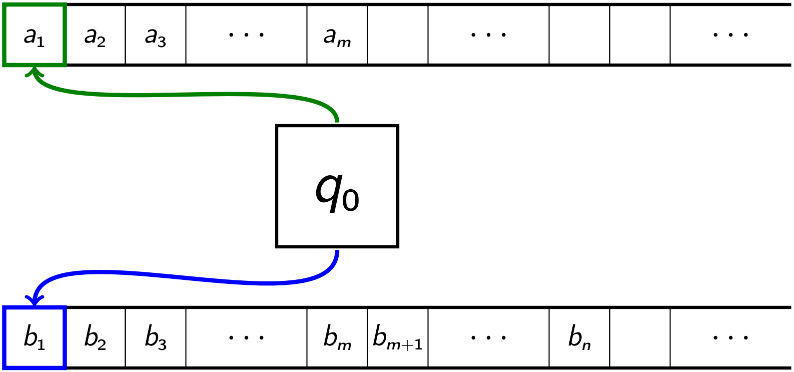

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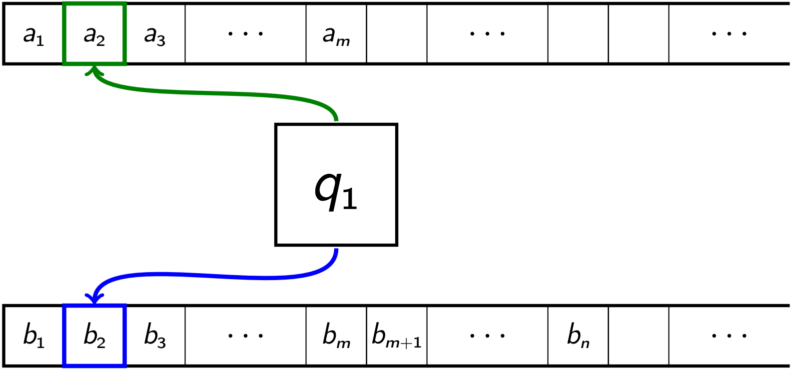

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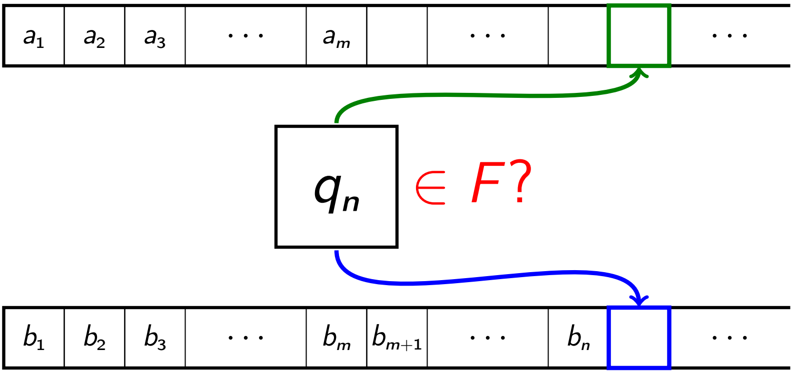

Convolution of words and regular word relations

Definition

Let $\square \notin \Sigma$ and $\Sigma_\square = \Sigma \cup \{\square\}$. For n words $w_1, \ldots, w_n \in \Sigma^\star$ their **convolution**

$$\otimes(w_1, \ldots, w_n) \in \left(\Sigma_\square^n\right)^\star$$

is defined as illustrated below.

Convolution of three words for $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$:

$$\otimes(\mathtt{aabca}, \mathtt{baccba}, \mathtt{cb}) = \boxed{\begin{matrix}\mathtt{a}\\ \mathtt{b}\\ \mathtt{c}\end{matrix}}\,\boxed{\begin{matrix}\mathtt{a}\\ \mathtt{a}\\ \mathtt{b}\end{matrix}}\,\boxed{\begin{matrix}\mathtt{b}\\ \mathtt{c}\\ \square\end{matrix}}\,\boxed{\begin{matrix}\mathtt{c}\\ \mathtt{c}\\ \square\end{matrix}}\,\boxed{\begin{matrix}\mathtt{a}\\ \mathtt{b}\\ \square\end{matrix}}\,\boxed{\begin{matrix}\square\\ \mathtt{a}\\ \square\end{matrix}} \in \left(\Sigma_\square^3\right)^\star$$

Elements of $\Sigma_\square^3$

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Definition

An n-ary word relation $R \subseteq (\Sigma^\star)^n$ is **automatic** if the word language

$$\otimes R = \{\ \otimes(w_1, \ldots, w_n) \mid (w_1, \ldots, w_n) \in R\ \} \subseteq \left(\Sigma_\square^n\right)^\star$$

is regular.

Word automatic structures

Definition

A relational structure $\mathcal{S} = (U; R_1, \ldots, R_n)$ is **word automatic** if there are

- an alphabet Σ and
- an injective mapping $h\colon U \to \Sigma^\star$ (the naming function)

such that

- $h(U)$ is a regular word language and
- $h(R_1), \ldots, h(R_n)$ are automatic word relations.

Notice: The structure $\big(h(U); h(R_1), \ldots, h(R_n)\big)$ is isomorphic to $\mathcal{S}$.

A tuple $\overline{\mathbf{M}} = (\mathbf{M}; \mathbf{M}_1, \ldots, \mathbf{M}_n)$ consisting of

- a finite automaton $\mathbf{M}$ accepting $h(U)$ and
- finite automata $\mathbf{M}_i$ accepting $\otimes h(R_i)$ for each $i = 1, \ldots, n$

is a **word automatic presentation** of $\mathcal{S}$.

The fundamental theorem on word automatic structures

Theorem (Khoussainov, Nerode 1995)

Let $\mathcal{S}$ be a word automatic structure and R a first order definable relation on $\mathcal{S}$.

1. *The structure $\mathcal{S}' = (\mathcal{S}; R)$ is also word automatic.*
2. *One can compute a word automatic presentation of $\mathcal{S}'$ from a word automatic presentation of $\mathcal{S}$ and a first order formula defining R.*

Corollary (Khoussainov, Nerode 1995)

The first order theory of a word automatic structure is decidable.

Examples of word automatic structures

The following structures are word automatic:

- finite structures
- $(\mathbb{N}; +)$, $(\mathbb{Z}; +)$, $(\mathbb{N}; <)$, $(\mathbb{Z}; <)$, $(\mathbb{Q}; <)$
- ordinals $< \omega^\omega$
- Caley graphs of many interesting groups
- configuration graphs of Turing machines ($\Rightarrow$ undecidable MSO-theory)

The following structures are **not** word automatic:

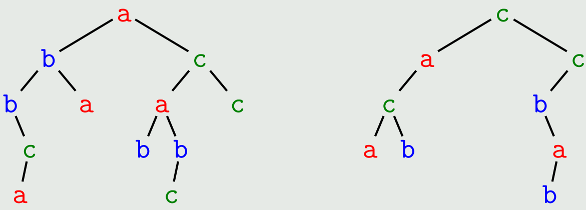
- $(\mathbb{N}; \times)$, $(\mathbb{Q}; +)$
- ordinals $\geq \omega^\omega$
- infinite fields
- the random graph (a.k.a. Rado graph)
- uncountable structures
- structures with undecidable first order theory

Trees

Definition

Let Σ be an alphabet. A **Σ-tree**, or just **tree**, is a map $t\colon D \to \Sigma$, where $D = \mathrm{dom}(t)$ is a non-empty, finite, and prefix-closed subset of $\{0,1\}^\star$.
The set of all Σ-tree is denoted with T_Σ.

Two Σ trees for $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$:



Tree automata and regular tree languages

Definition

A (deterministic bottom-up) **tree automaton** $\mathbf{A} = (Q, q_0, \delta, F)$ (over Σ) consists of

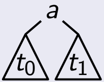
- a non-empty, finite set of states Q
- an initial state $q_0 \in Q$
- a transition function $\delta \colon Q \times Q \times \Sigma \to Q$
- a set $F \subseteq Q$ of accepting states

The state which is reached (at the root) after processing a tree $t \in T_\Sigma$ bottom-up is denoted with $\mathbf{A}(t)$.

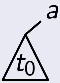
Bottom-up processing of trees

Definition

The state which is reached (at the root) after a tree automaton $\mathbf{A} = (Q, q_0, \delta, F)$ processed a tree $t \in T_\Sigma$ bottom-up is denoted with $\mathbf{A}(t)$ and defined inductively on the structure of t:

1. $t =$ 

: $\mathbf{A}(t) = \delta\big(\mathbf{A}(t_0), \mathbf{A}(t_1), a\big)$

2. $t =$ 

: $\mathbf{A}(t) = \delta\big(\mathbf{A}(t_0), q_0, a\big)$

3. $t =$ 

: $\mathbf{A}(t) = \delta\big(q_0, \mathbf{A}(t_1), a\big)$

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A (deterministic bottom-up) **tree automaton** $\mathbf{A} = (Q, q_0, \delta, F)$ (over Σ) consists of

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- a set $F \subseteq Q$ of accepting states

The state which is reached (at the root) after processing a tree $t \in T_\Sigma$ bottom-up is denoted with $\mathbf{A}(t)$.
The tree language **accepted** by $\mathbf{A}$ is

$$L(\mathbf{A}) = \{\, t \in T_\Sigma \mid \mathbf{A}(t) \in F \,\}\,.$$

Definition

A tree language $L \subseteq T_\Sigma$ is **regular** if it can be accepted by some tree automaton.

Convolution of trees and regular tree relations

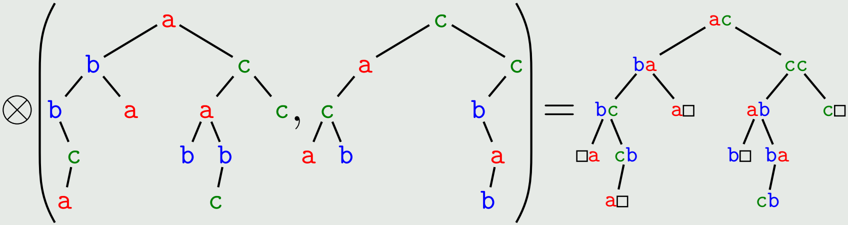
Definition

Let $\square \notin \Sigma$ and $\Sigma_\square = \Sigma \cup \{\square\}$. For n trees $t_1, \ldots, t_n \in T_\Sigma$ their **convolution**

$$\otimes(t_1, \ldots, t_n) \in T_{\Sigma_\square^n}$$

is defined as illustrated below.

Convolution of two trees for $\Sigma = \{a, b, c\}$:



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Let $\square \notin \Sigma$ and $\Sigma_\square = \Sigma \cup \{\square\}$. For n trees $t_1, \ldots, t_n \in T_\Sigma$ their **convolution**

$$\otimes(t_1, \ldots, t_n) \in T_{\Sigma_\square^n}$$

is defined as illustrated below.

Definition

An n-ary tree relation $R \subseteq \left(T_\Sigma\right)^n$ is **automatic** if the tree language

$$\otimes R = \{\, \otimes(t_1, \ldots, t_n) \mid (t_1, \ldots, t_n) \in R \,\} \subseteq T_{\Sigma_\square^n}$$

is regular.

Tree automatic structures

Definition

The notions of **tree automatic** and **tree automatic presentation** are defined analogously to the word case.

Theorem (Blumensath 1999)

Let $\mathcal{S}$ be a tree automatic structure and R a first order definable relation on $\mathcal{S}$.

1. *The structure $\mathcal{S}' = (\mathcal{S}; R)$ is also tree automatic.*
2. *One can compute a tree automatic presentation of $\mathcal{S}'$ from a tree automatic presentation of $\mathcal{S}$ and a first order formula defining R.*

Corollary (Blumensath 1999)

The first order theory of a tree automatic structure is decidable.

Examples of tree automatic structures

The following structures are tree automatic:

- word automatic structures
- $(\mathbb{N}; \times)$, $(\mathbb{Z}; \times)$, $(\mathbb{Q}; \times)$
- ordinals $< \omega^{\omega^\omega}$

The following structures are **not** tree automatic:

- ordinals $\geq \omega^{\omega^\omega}$
- the random graph (a.k.a. Rado graph)
- uncountable structures
- structures with undecidable first order theory

Motivation and main result

Observation

Every word automatic structure is also tree automatic.

Problem

Given a tree automatic presentation of some structure (from a certain class of structures), is it decidable whether this structure is already word automatic?

As far as we know, there are no (interesting) classes of structures for which this question has been answered, positively or negatively, yet.

Theorem (H 2011)

1. *Given a tree automatic presentation of an ordinal α, it is decidable whether α is word automatic*
2. *In case α is word automatic, one can compute a word automatic presentation of α from the tree automatic presentation.*

Proof sketch of the main result

Theorem (H 2011)

1. *Given a tree automatic presentation of an ordinal α, it is decidable whether α is word automatic*
2. *In case α is word automatic, one can compute a word automatic presentation of α from the tree automatic presentation.*

Proof sketch.

We introduce the notion of **slim** tree automatic presentations and show this property to be:

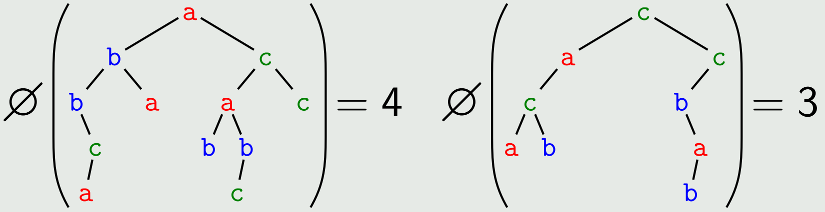
1. **decidable**,
2. **sufficient** for being word automatic for any kind of structure, and
3. **necessary** for being word automatic in case of **ordinals**. $\triangle$

The level size

Definition

The **thickness** of a tree $t \in T_\Sigma$ is

$$\varnothing(t) = \max\{\ |\mathrm{dom}(t) \cap \{0,1\}^\ell|\ \big|\ \ell \geq 0\ \} \in \mathbb{N}\,.$$



The level size

Definition

The **thickness** of a tree $t \in T_\Sigma$ is

$$\varnothing(t) = \max\left\{ \, |\mathrm{dom}(t) \cap \{0,1\}^\ell| \;\middle|\; \ell \geq 0 \, \right\} \in \mathbb{N}\,.$$

The **thickness** of a tree language $L \subseteq T_\Sigma$ is

$$\varnothing(L) = \sup\left\{ \, \varnothing(t) \;\middle|\; t \in L \, \right\} \in \mathbb{N} \cup \{\infty\}\,.$$

If $\varnothing(L) < \infty$, then L is **slim**, otherwise L is **fat**.

Definition

A tree automatic presentation $(\mathbf{A}; \dots)$ of some structure is **slim** (resp. **fat**) if $L(\mathbf{A})$ is slim (resp. fat).

Proposition 1

Proposition 1

1. *Given a tree automatic presentation* $\overline{\mathbf{A}} = (\mathbf{A}; \ldots)$, *it is decidable whether* $\overline{\mathbf{A}}$ *is slim or fat.*
2. *In case* $\overline{\mathbf{A}}$ *is slim, the thickness of* $L(\mathbf{A})$ *is at most* $(n+1) \cdot 2^n$, *where* n *is the number of states of* $\mathbf{A}$.

Proof.

Similar to deciding whether $L(\mathbf{A})$ is a finite or an infinite set, but a bit more involved. $\triangle$

Proposition 2

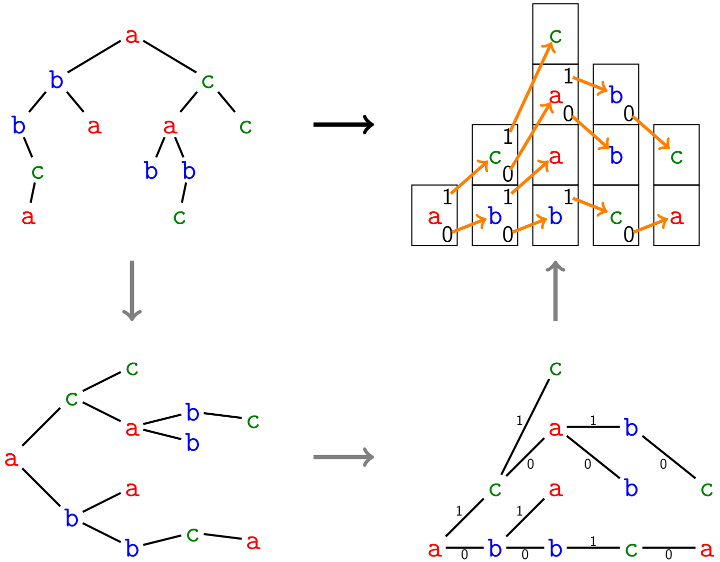
Proposition 2

1. *Every structure $\mathcal{S}$ that admits a slim tree automatic presentation is word automatic.*
2. *One can compute a word automatic presentation of $\mathcal{S}$ from a slim tree automatic presentation.*

Proof sketch.

- Encode trees by words.
- Construct (non-deterministic) finite automata which simulate the automata of the slim tree automatic presentation on these encodings. □

Encoding trees by words



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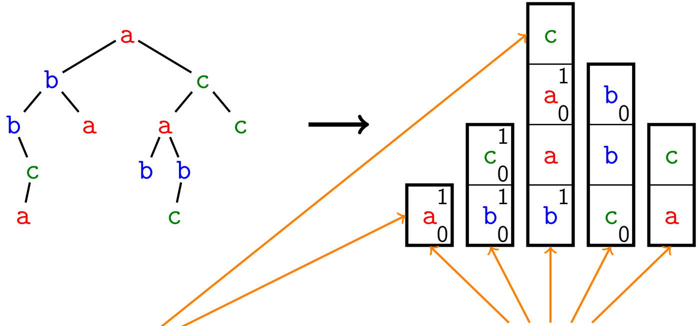
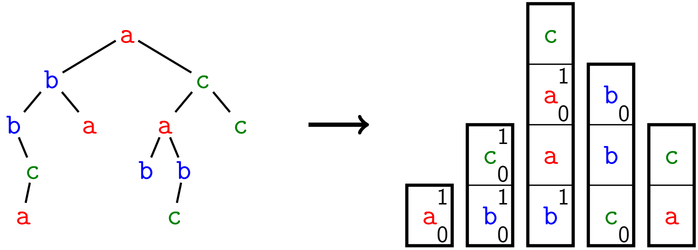


Elements of $\Sigma \times 2^{\{0,1\}}$

Elements of $\bigcup_{1 \le \ell \le 4} \left(\Sigma \times 2^{\{0,1\}}\right)^{\ell}$

Encoding trees by words



Definition

A tree $t \in T_\Sigma$ of thickness k is encoded by a word

$$C(t) \in \Gamma_k^\star$$

over the alphabet

$$\Gamma_k = \bigcup_{1 \leq \ell \leq k} \left(\Sigma \times 2^{\{0,1\}} \right)^\ell .$$

Encoding tree languages by word languages

Definition

A tree language $L \subseteq T_\Sigma$ of thickness at most k is encoded by the word language

$$C(L) = \left\{ C(t) \;\middle|\; t \in L \right\} \subseteq \Gamma_k^\star .$$

Lemma

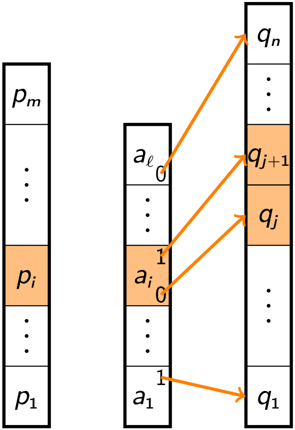
1. *If L is a regular, then $C(L)$ is regular.*
2. *One can compute a (non-deterministic) finite automaton accepting $C(L)$ from k and a tree automaton accepting L.*

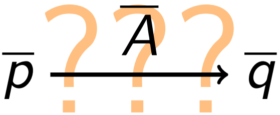
Proof.

Let $\mathbf{A} = (Q, q_0, \delta, F)$ be a tree automaton accepting L. We construct a non-deterministic finite automaton accepting $C(L)$ with state space

$$\bigcup_{0 \leq m \leq k} Q^m .$$

The non-deterministic finite automaton for $C(L)$





The transition $\overline{p} \xrightarrow{\overline{A}} \overline{q}$ exists precisely if $m = \ell$, n "fits" $\overline{A}$, and for all $i = 1, \ldots, \ell$ one of the four conditions is met:

1. $A_i = \boxed{{a_i}^{1}_{0}}$ and $p_i = \delta(q_j, q_{j+1}, a_i)$
2. $A_i = \boxed{{a_i}_{0}}$ and $p_i = \delta(q_j, q_0, a_i)$
3. $A_i = \boxed{{a_i}^{1}}$ and $p_i = \delta(q_0, q_{j+1}, a_i)$
4. $A_i = \boxed{a_i}$ and $p_i = \delta(q_0, q_0, a_i)$

where j is suitable (like in the picture).

Initial are the states $\boxed{f}$ for $f \in F$.

Accepting is only the single state from Q^0.

Encoding of tree relations by word relations

Definition

Let $L \subseteq T_\Sigma$ be a slim tree language. An n-ary tree relation $R \subseteq L^n$ is encoded by the word relation

$$C(R) = \{ \, (C(t_1), \ldots, C(t_n)) \mid (t_1, \ldots, t_n) \in R \, \} \subseteq C(L)^n \, .$$

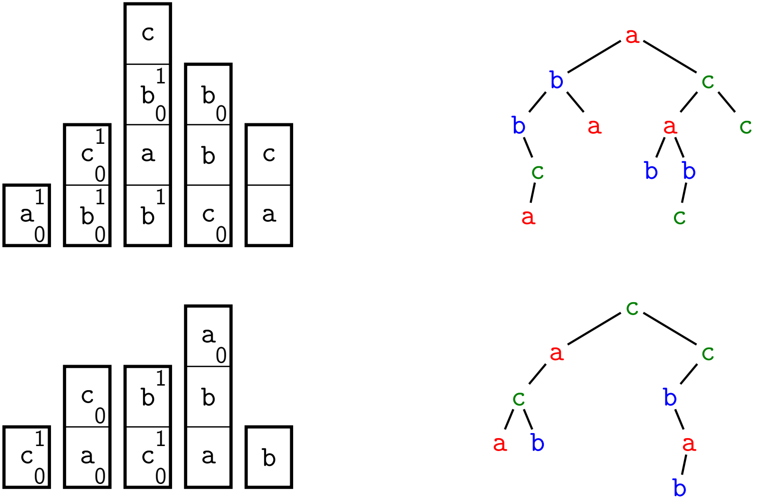
Lemma

1. *If R is automatic, then $C(R)$ is automatic.*
2. *One can construct a (non-deterministic) finite automaton accepting $\otimes C(R)$ from k and a tree automaton accepting $\otimes R$.*

Proof.

Similar to the proof for tree languages, but a bit more involved.

The second key idea behind the finite automaton for $\otimes C(R)$



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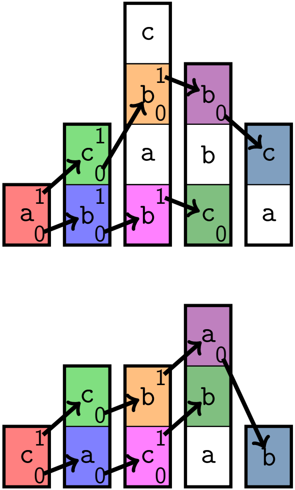
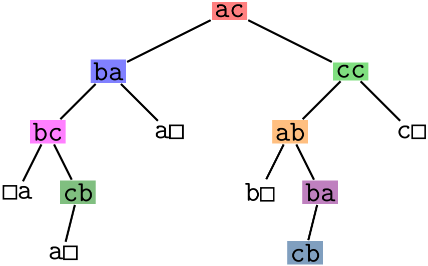




Proposition 3

Proposition 3

If an ordinal α admits a ***fat*** *tree automatic presentation, then α is* ***not*** *word automatic.*

Theorem (Delhommé 2001)

An ordinal α is word automatic if, and only if, $\alpha < \omega^\omega$.

Proposition 3'

If an ordinal α admits a ***fat*** *tree automatic presentation, then*

$$\alpha \geq \omega^\omega .$$

Proposition 3'

Proposition 3'

*If an ordinal α admits a **fat** tree automatic presentation $(\mathbf{A}; \mathbf{A}_<)$, then*

$$\alpha \geq \omega^\omega .$$

Lemma

Let $n, r \geq 1$ and $(\mathbf{A}; \mathbf{A}_<)$ be a tree automatic presentation of some ordinal α such that $\mathbf{A}$ has n states and the thickness of $L(\mathbf{A})$ is at least $r \cdot 2^n$. Then,

$$\alpha \geq \omega^r .$$

Proof of Proposition 3'.

Since $L(\mathbf{A})$ has thickness ∞, we have $\alpha \geq \omega^r$ for all $r \geq 1$. Thus,

$$\alpha \geq \sup\{\, \omega^r \mid r \geq 1 \,\} = \omega^\omega . \qquad \square$$

The key lemma

Lemma

Let $n, r \geq 1$ and $(\mathbf{A}; \mathbf{A}_<)$ be a tree automatic presentation of some ordinal α such that $\mathbf{A}$ has n states and the thickness of $L(\mathbf{A})$ is at least $r \cdot 2^n$. Then,

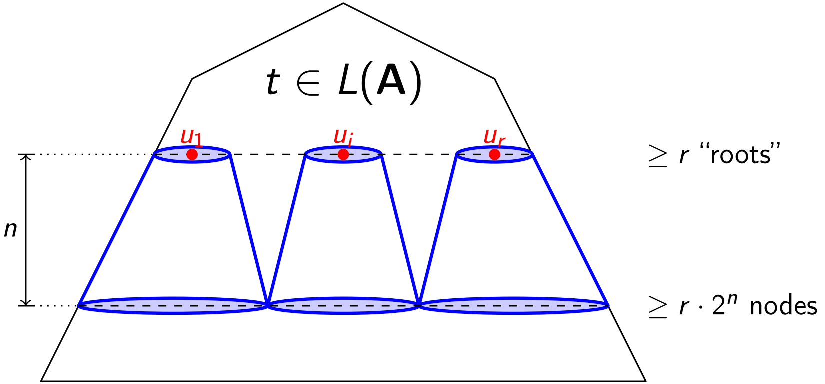
$$\alpha \geq \omega^r \, .$$

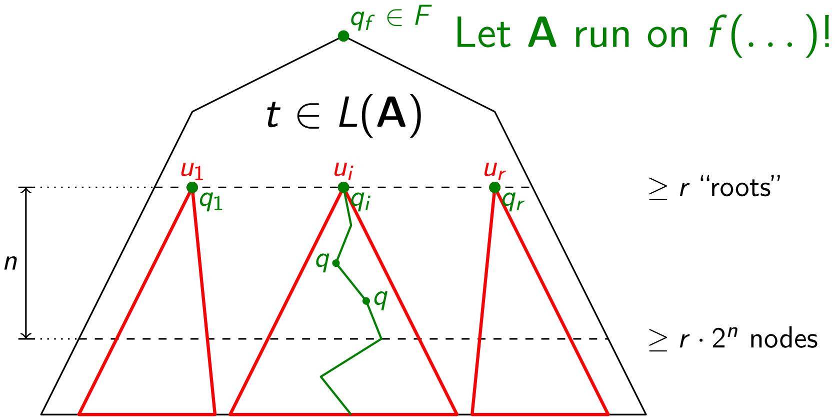
Proof sketch.

- Let $<$ be the order on $L(\mathbf{A})$ recognized by $\mathbf{A}_<$, i.e.,

$$s < t \iff s \otimes t \in L(\mathbf{A}_<) \, .$$

- Consider a tree $t \in L(\mathbf{A})$ with thickness at least $r \cdot 2^n$.
- Starting with t, construct a subset of $L(\mathbf{A})$ which has order type (w.r.t. $<$) at least α.

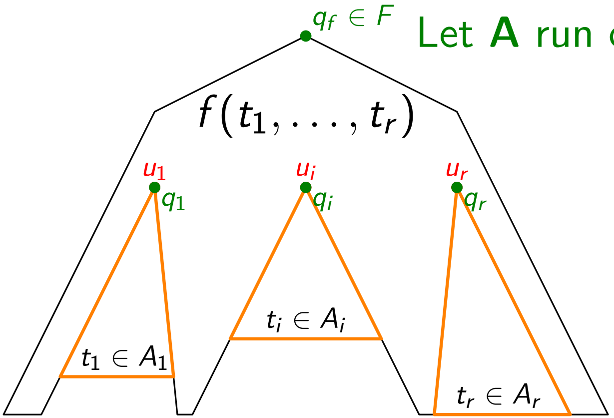

$t \in L(\mathbf{A})$
u_1
u_i
u_r
n
$\geq r$ “roots”
$\geq r \cdot 2^n$ nodes



Observations

- the set $A_i = \{ t_i \in T_\Sigma \mid \delta(t_i) = q_i \}$ is infinite

Let **A** run on $f(\dots)$!



Observations

- the set $A_i = \{ t_i \in T_\Sigma \mid \delta(t_i) = q_i \}$ is infinite
- $f(t_1, \dots, t_r) \in L(\mathbf{A})$ for all $t_1 \in A_1, \dots, t_r \in A_r$
- $\mathbf{A}_<\big(f(s_1, \dots, s_r) \otimes f(t_1, \dots, t_r)\big)$ is determined by the $\mathbf{A}_<(s_i \otimes t_i)$

- By Ramsey's theorem (for infinite undirected colored graphs) there is an infinite subset $B_i \subseteq A_i$ such that

$$C(s_i, t_i) = \{\mathbf{A}_<(s_i \otimes s_i), \mathbf{A}_<(t_i \otimes t_i), \mathbf{A}_<(s_i \otimes t_i), \mathbf{A}_<(t_i \otimes s_i)\}$$

is the same set C_i for all $s_i, t_i \in B_i$ with $s_i \neq t_i$.

- The set C_i has exactly three elements, say $q_i^=, q_i^<, q_i^>$, which satisfy for $s_i, t_i \in B_i$:

$$f(\alpha, s_i, \beta) < f(\alpha, t_i, \beta) \iff \mathbf{A}_<(s_i \otimes t_i) = q_i^< \;:\!\iff\; s_i <_i t_i$$

$$f(\alpha, s_i, \beta) = f(\alpha, t_i, \beta) \iff \mathbf{A}_<(s_i \otimes t_i) = q_i^= \iff s_i = t_i$$

$$f(\alpha, s_i, \beta) > f(\alpha, t_i, \beta) \iff \mathbf{A}_<(s_i \otimes t_i) = q_i^> \iff s_i >_i t_i$$

where $\alpha \in B_1 \times \cdots \times B_{i-1}$ and $\beta \in B_{i+1} \times \cdots \times B_r$ are arbitary.

- The relation $<_i$ is a well-ordering on B_i.

- The relation $<_i$ defined by

$$s_i <_i t_i \;:\Longleftrightarrow\; f(\alpha, s_i, \beta) < f(\alpha, t_i, \beta)$$

  is a well-ordering on B_i.
- Let $\mathbf{0}_i$ and $\mathbf{1}_i$ be the least and the second least (w.r.t. $<_i$) element of B_i and put

$$e_i = f(\mathbf{0}_1, \ldots, \mathbf{0}_{i-1}, \mathbf{1}_i, \mathbf{0}_{i+1}, \ldots, \mathbf{0}_r)\,.$$

- W.l.o.g. we assume $e_1 > e_2 > \cdots > e_r$.
- Let $<_{\mathrm{lex}}$ be the lexicographic ordering on $B_1 \times \cdots \times B_r$. The map

$$f\colon B_1 \times \cdots \times B_r \to L(\mathbf{A})$$

  is order preserving embedding (w.r.t. $<_{\mathrm{lex}}$ and $<$).
- Let $\beta_i \geq \omega$ be the order type of B_i (w.r.t $<_i$). Then

$$\alpha \geq \beta_r \cdots \beta_1 \geq \omega \cdots \omega = \omega^r\,. \qquad \square$$

Summary and generalization

Theorem (H 2011)

Given a tree automatic presentation of an ordinal α, it is decidable whether α is word automatic.

Theorem (H 2011)

Given a tree automatic presentation of a scattered linear ordering $\mathcal{L}$, it is decidable whether $\mathcal{L}$ is word automatic.

Theorem (Delhommé 2001)

An ordinal α is word automatic if, and only if, $\alpha < \omega^\omega$.

Theorem (KRS 2003)

The Hausdorff rank of every word automatic scattered linear ordering is finite.

Outlook

Open questions

- Given a tree automatic presentation of a linear ordering $\mathcal{L}$, is it decidable whether $\mathcal{L}$ is word automatic?
  (New techniques are necessary, since $(\mathbb{Q}; <)$ is a word automatic **and** admits a **fat** tree automatic presentation.)
- What about other classes of structures, e.g., Boolean algebras?

Another research direction

- For every tree automatic ordinal α the set of all (names of) ω-limit points of α is (effectively) regular.
- Are the sets of all (names of) ω^n-limit points for each $n \geq 1$ also (effectively) regular?
- If they are, we can show that the isomorphism problem for tree automatic ordinals is decidable.