Isomorphism of regular trees and words

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A generalized word is regular iff it is the smallest solution of a system of equations.

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Observation

A generalized word w is regular $\iff w = \text{yield}(T)$ for a regular tree T.

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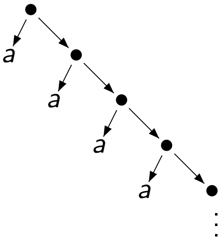

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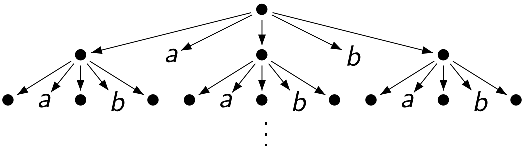

Regular words (2nd alternative characterization)

A partitioned DFA is a tuple

$$A = (Q, \{1, \ldots, k\}, \delta, q_0, (F_a)_{a \in \Sigma}),$$

where Σ is a finite alphabet an $(Q, \{1, \ldots, k\}, \delta, q_0, \bigcup_{a \in \Sigma} F_a)$ is an ordinary DFA.

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Heilbrunner 1980

A generalized word w is regular $\iff$ $w = \mathrm{val}(e)$ for a regular expression e.

Isomorphism problem for a regular words I

Thomas 1986

It is decidable, whether $w_1 \cong w_2$ for two given regular words w_1, w_2 (given, e.g., by regular expressions or partitioned DFAs).

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Bloom, Esik 2005

It can be checked in in polynomial time, whether $\mathrm{val}(e_1) \cong \mathrm{val}(e_2)$ for two given regular expressions e_1, e_2.

Isomorphism problem for a regular II

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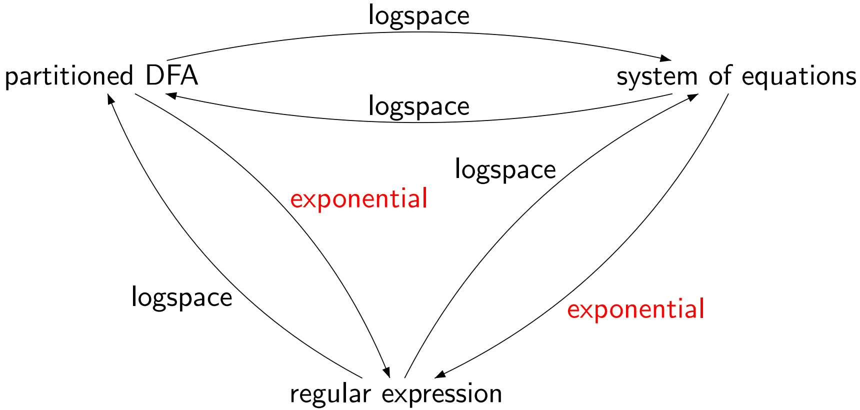
Complexity of transforming representations for regular words:



Our main result

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Partitioned DFA $\longrightarrow$ DAG

From a given partitioned DFA A one can compute in logspace a DAG (directed acyclic graph), whose unfolding is a regular expression e with $\mathsf{val}(e) = w(A)$.

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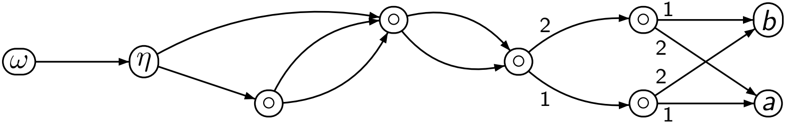
Proposition

It can be checked in polynomial time whether for two given DAGs, the corresponding regular expressions define isomorphic regular words.

DAGs: An example



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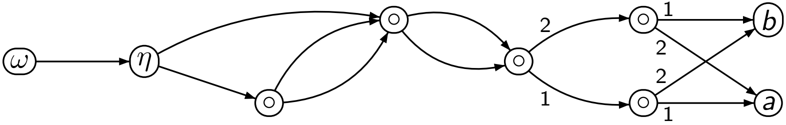

This DAG produces the generalized word

$$[abbaabba, abbaabbaabbaabba]^{\eta})^{\omega}.$$

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But: We reduce (by a polyomial time Turing reduction) the equivalence problem for general DAGs to those for SLPs, following the strategy of Bloom and Esik.

Related results

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It is PSPACE-hard (and in EXPTIME) to check for two given NFAs A_1, A_2, whether $(L(A_1), \leq_{\text{lex}}) \cong (L(A_2), \leq_{\text{lex}})$.

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