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First-order and counting theories of ω-automatic structures

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Abstract. The logic $\mathcal{L}(\mathcal{Q}_u)$ extends first-order logic by a generalized form of counting quantifiers ("the number of elements satisfying ... belongs to the set C"). This logic is investigated for structures with an injective ω-automatic presentation. If first-order logic is extended by an infinity-quantifier, the resulting theory of any such structure is known to be decidable [4]. It is shown that, as in the case of automatic structures [13], also modulo-counting quantifiers as well as infinite cardinality quantifiers ("there are $\varkappa$ many elements satisfying ...") lead to decidable theories. For a structure of bounded degree with injective ω-automatic presentation, the fragment of $\mathcal{L}(\mathcal{Q}_u)$ that contains only effective quantifiers is shown to be decidable and an elementary algorithm for this decision is presented. Both assumptions (ω-automaticity and bounded degree) are necessary for this result to hold.

1 Introduction

Automatic structures were introduced in [8, 11]. The idea goes back to the concept of automatic groups [6]. Roughly speaking, a structure is called automatic if the elements of the universe are represented (not necessarily uniquely) as words from a regular language and every relation (including the identity) of the structure can be recognized by a finite state automaton with several heads that proceed synchronously. Automatic structures received increasing interest during the last years [1, 4, 9, 12, 14, 16]. Recently, automatic structures were generalized to ω-automatic structures by the use of Büchi-automata instead of automata on finite words [4]. One of the main motivations for investigating (ω-)automatic structures is the fact that every (ω-)automatic structure has a decidable first-order theory [4, 11]. For automatic structures, this result has been extended to first-order logic with modulo quantifiers [13] and the quantifier "there exist infinitely many" (infinity quantifier) [4]. The infinity quantifier was also shown to lead to decidable theories in the realm of ω-automatic structures [3, 4] with injective presentations (i.e., if the elements of the structure are represented by unique ω-words).[3] While there exist automatic structures with a non-elementary first-order theory [4], the first-order theory of any automatic structure of bounded

[3] The decidability proof of [4, Thm. 2.1] assumes an injective ω-automatic presentation. [4, Prop. 5.2] states that any ω-automatic structure has such an injective

degree is elementarily decidable; more precisely, an upper bound of triply exponential alternating time with a linear number of alternations was shown in [16].

The overall theme of this paper is to extend these results from automatic structures to ω-automatic structures and to consider more involved logics. In a *first step*, we extend first-order logic by modulo-counting quantifiers as in [13] and exact counting quantifiers for infinite cardinals. We show that any injectively ω-automatic structure has a decidable theory in this logic (Theorem 2.8). This extends [13, Theorem 3.2] from automatic to injectively ω-automatic structures and [4, Theorem 2.1] from first-order logic with an infinity quantifier to a further extension of this logic. The proof is based on automata-theoretic constructions, in particular an analysis of successful runs in Muller automata.

In a *second step*, we consider an even more powerful logic that we call $\mathcal{L}(\mathcal{Q}_u)$, which is a finitary fragment of the logic $\mathcal{L}_{\infty,\omega}(\mathcal{Q}_u)^\omega$ from [10]. In this logic $\mathcal{L}(\mathcal{Q}_u)$ one may use generalized quantifiers of the form $\mathcal{Q}_\mathcal{C} y : (\psi_1(y), \ldots, \psi_n(y))$, where y is a first-order variable and $\mathcal{C}$ is an n-ary relation on cardinals. To determine the truth of this formula in a model $\mathcal{A}$, one first determines the cardinalities of the sets defined by the formulas $\psi_i(y)$ $(1 \leq i \leq n)$. If the tuple of these cardinalities belongs to the relation $\mathcal{C}$, then the formula is true. All quantifiers mentioned so far are special instances of these generalized quantifiers. But, e.g., also the Härtig quantifier ("there are as many ... as ...") falls into this category.

For every fragment $\mathcal{L}$ of $\mathcal{L}(\mathcal{Q}_u)$ that contains only countably many generalized quantifiers, and every injectively ω-automatic structure $\mathcal{A}$ of bounded degree, we prove that the $\mathcal{L}$-theory of $\mathcal{A}$ can be decided by a Turing-machine with oracle access to the relations $\mathcal{C}$ that are allowed in the fragment $\mathcal{L}$. Moreover, this Turing-machine works in triply exponential space (Theorem 3.7). This extends [16, Theorem 3] since it applies to (1) injectively ω-automatic structures as opposed to automatic structures and (2) to first-order logic extended by generalized quantifiers. This second main result rests on [10] where Hanf-locality is shown for the logic $\mathcal{L}(\mathcal{Q}_u)$. Our algorithm therefore has to determine how often a given neighborhood is realized (up to isomorphism) in the structure. Differently, in the proof of [16, Theorem 3] a similar locality principle is used to effectively bound the search space of quantifiers to short words.

From Theorem 3.7 we deduce that every $\mathcal{L}$-definable relation over an injectively ω-automatic structure of bounded degree is *effectively* first-order definable and therefore *effectively* regular (Corollary 3.9). If effectiveness is not demanded, first-order definability can be easily deduced also for non-ω-automatic structures of bounded degree from [10].

Note that our results require a structure to be ω-automatic and of bounded degree. We finish the technical part of the paper by showing that both these assumptions are necessary, namely that our results do not hold for recursive structures of bounded degree, nor for locally finite automatic (and hence locally finite injectively ω-automatic) structures.

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presentation, but the proof is spurious (cf. Remark 2.1). So we safely use the decidability for injective presentations, only.

Proofs that are omitted due to space restrictions can be found in the technical report [15].

2 ω-automatic structures, infinity and modulo quantifiers

2.1 Definitions and known results

This section introduces automata on finite and on infinite words, (ω-)automatic structures, and logics, and recalls some basic results concerning these concepts. For more details, see [17, 18] for automata theoretic issues, [4, 11, 13] for ω-automatic structures, and [7] as far as logics are concerned.

Büchi-automata. Let Γ be a finite alphabet. With Γ^* we denote the set of all finite words over the alphabet Γ. The set of all nonempty finite words is Γ^+. An *ω-word* over Γ is an infinite ω-sequence $w = a_0a_1a_2\cdots$ with $a_i \in \Gamma$, we set $w(i) = a_i$ for $i \in \mathbb{N}$. A (nondeterministic) *Büchi-automaton* M is a tuple $M = (Q, \Gamma, \delta, \iota, F)$, where Q is a finite set of states, $\iota \in Q$ is the initial state, $F \subseteq Q$ is the set of final states, and $\delta \subseteq Q \times \Gamma \times Q$ is the transition relation. If $\Gamma = \Sigma^n$ for some alphabet Σ, then we speak of an *n-dimensional Büchi-automaton over Σ*. A *run* of M on an ω-word $w = a_0a_1a_2\cdots$ is an ω-word $r = p_0p_1p_2\cdots$ over the set of states Q such that $(p_i, a_i, p_{i+1}) \in \delta$ for all $i \geq 0$. The run r is *successful* if $p_0 = \iota$ and there exists a final state from F that occurs infinitely often in r. The language $L_\omega(M) \subseteq \Gamma^\omega$ defined by M is the set of all ω-words for which there exists a successful run. An ω-language $L \subseteq \Gamma^\omega$ is *regular* if there exists a Büchi-automaton M with $L_\omega(M) = L$. The class of all regular ω-languages is closed under boolean operations and projections [17]. For two Büchi-automata M_1 and M_2 with n_1 and n_2 many states, resp., there exists a Büchi-automaton with $3 \cdot n_1 \cdot n_2$ many states accepting the language $L_\omega(M_1) \cap L_\omega(M_2)$. The proof is based on a product construction for Büchi-automata, see e.g. [18]. For ω-words $w_1, \ldots, w_n \in \Gamma^\omega$, the *convolution* $w_1 \otimes w_2 \otimes \cdots \otimes w_n \in (\Gamma^n)^\omega$ is

$$w_1 \otimes \cdots \otimes w_n = (w_1(1), \ldots, w_n(1))\,(w_1(2), \ldots, w_n(2))\,(w_1(3), \ldots, w_n(3))\cdots$$

An n-ary relation $R \subseteq (\Gamma^\omega)^n$ is called *ω-automatic* if the language $\{w_1 \otimes \cdots \otimes w_n \mid (w_1, \ldots, w_n) \in R\}$ is a regular ω-language, i.e., accepted by some n-dimensional Büchi-automaton.

ω-automatic structures. A *signature* is a finite set τ of relational symbols, where each relational symbol $R \in \tau$ has an associated arity n_R. A (relational) *structure over the signature τ*, briefly a *τ-structure*, is a tuple $\mathcal{A} = (A, (R^{\mathcal{A}})_{R\in\tau})$, where A is a set (the universe of $\mathcal{A}$) and $R^{\mathcal{A}}$ is a relation of arity n_R over the set A, which interprets the relational symbol R. We will assume that every signature contains the equality symbol $=$ and that $=^{\mathcal{A}}$ is the identity relation on the set A. Usually, we denote the relation $R^{\mathcal{A}}$ also with R. We will also write $a \in \mathcal{A}$ for $a \in A$. For a subset $B \subseteq A$ we denote with $\mathcal{A}{\restriction}B$ the restriction $(B, (R^{\mathcal{A}} \cap B^{n_R})_{R\in\tau})$.

Let $\mathcal{A}$ be an arbitrary τ-structure with universe A. An *injectively ω-automatic presentation* for $\mathcal{A}$ is a tuple (Γ, L, h) such that

- Γ is a finite alphabet,
- $L \subseteq \Gamma^\omega$ is a regular ω-language,
- $h : L \to A$ is a bijection, and
- the relation $\{(u_1, \ldots, u_{n_R}) \in L^{n_R} \mid (h(u_1), \ldots, h(u_{n_R})) \in R\}$ is ω-automatic for every $R \in \tau$.

The structure $\mathcal{A}$ is injectively *ω-automatic* if there is an injectively ω-automatic presentation for $\mathcal{A}$. A typical example of an injectively ω-automatic structure is $(\mathbb{R}, +)$.

Remark 2.1. The original definition of an ω-automatic presentation requires h to be only surjective and the relation $\{(u, v) \in L^2 \mid h(u) = h(v)\}$ to be ω-automatic [4]. In [4, Proposition 5.2] it is claimed that every ω-automatic structure (according to this original definition) has an injectively ω-automatic presentation. The following example shows that the proof of [4, Proposition 5.2] does not work: Let two sets A and B of natural numbers be equivalent ($A \approx B$) if and only if the symmetric difference $A \triangle B$ is finite. Then the quotient $\mathcal{B}$ of the power-set of $\mathbb{N}$ wrt. $\approx$ is a Boolean algebra. It has an ω-automatic presentation in the more general sense of [4] with underlying set $L = \{0, 1\}^\omega$ and $h(w) = [\{i \in \mathbb{N} \mid w(i) = 1\}]_\approx$. But there is no ω-regular subset $K \subseteq L$ such that, for any $u \in L$, there is precisely one $v \in K$ with $h(u) = h(v)$, as was claimed in [4]. It is therefore open, whether every ω-automatic structure (in the original sense) has an injectively ω-automatic presentation. Since this paper deals with injectively ω-automatic structures exclusively, we will always assume an injectively ω-automatic presentation (Γ, L, h), where L is the universe of the structure and h is the identity function. Furthermore, we use the more concise notation "ω-automatic presentation" (resp. "ω-automatic structure") instead of "injectively ω-automatic presentation" (resp. "injectively ω-automatic structure").

Automatic structures are defined in the same way as ω-automatic structures, except that finite automata over finite words instead of Büchi-automata are used (the convolution of finite words requires an additional letter $\perp$ that is appended to the arguments in order to make them the same length). By [3, Theorem 5.32], a countable structure is automatic if and only if it is ω-automatic.

Logic. In addition to the usual first-order quantifier $\exists$, this section is concerned with quantifiers $\exists^\infty$, $\exists^\varkappa$ for a cardinal $\varkappa$, and $\exists^{(t,k)}$ for $0 \leq t < k > 1$ two natural numbers. The semantics of these quantifiers are defined as follows:

- $\mathcal{A} \models \exists^\infty x\, \psi$ if and only if there are infinitely many $a \in \mathcal{A}$ with $\mathcal{A} \models \psi(a)$.
- $\mathcal{A} \models \exists^\varkappa x\, \psi$ if and only if the set $\{a \in \mathcal{A} \mid \mathcal{A} \models \psi(a)\}$ has cardinality $\varkappa$.
- $\mathcal{A} \models \exists^{(t,k)} x\, \psi$ if and only if the set $\{a \in \mathcal{A} \mid \mathcal{A} \models \psi(a)\}$ is finite and $t = |\{a \in \mathcal{A} \mid \mathcal{A} \models \psi(a)\}| \bmod k$.

We will denote by FO the set of first-order formulas. For a class of cardinals C, $\mathrm{FO}(\exists^\infty, (\exists^\varkappa)_{\varkappa\in C}, (\exists^{(t,k)})_{0\le t<k>1})$ is the set of formulas using $\exists$ and the quantifiers listed. For any set $\mathcal{L}$ of formulas, the *$\mathcal{L}$-theory* of a structure $\mathcal{A}$ is the set of sentences (i.e., formulas without free variables) from $\mathcal{L}$ that hold in $\mathcal{A}$. The following result can be shown by induction on the structure of the formula φ.

Proposition 2.2 (cf. [4, 11, 13]). *Let (Γ, L, h) be an automatic presentation for the structure $\mathcal{A}$ and let $\varphi(x_1, \dots, x_n)$ be a formula of* $\mathrm{FO}(\exists^\infty, (\exists^{(t,k)})_{0\le t<k\ge 2})$ *over the signature of $\mathcal{A}$. Then the relation*

$$\{(u_1, \dots, u_n) \in L^n \mid \mathcal{A} \models \varphi(h(u_1), \dots, h(u_n))\}$$

is effectively automatic. It is effectively ω-automatic if (Γ, L, h) is an ω-automatic presentation for the structure $\mathcal{A}$ and φ belongs to $\mathrm{FO}(\exists^\infty)$.

This theorem implies the following result, which is one of the main motivations for investigating (ω-)automatic structures.

Theorem 2.3 ([4, 13]). *If $\mathcal{A}$ is an ω-automatic structure, then its* $\mathrm{FO}(\exists^\infty)$*-theory is decidable. If $\mathcal{A}$ is automatic, then even its* $\mathrm{FO}(\exists^\infty, (\exists^{(t,k)})_{0\le t<k\ge 2})$*-theory is decidable.*

Note that any automatic structure $\mathcal{A}$ is at most countably infinite. Hence the quantifiers $\exists^\infty$ and $\exists^{\aleph_0}$ are equivalent in this setting. Furthermore, no formula $\exists^\varkappa x\, \psi$ with $\varkappa > \aleph_0$ holds in $\mathcal{A}$. Hence, for any countable set of cardinals C, the $\mathrm{FO}(\exists^\infty, (\exists^\varkappa)_{\varkappa\in C}, (\exists^{(t,k)})_{0\le t<k>1})$-theory of an automatic structure is decidable.[4] In the rest of Section 2 we extend this result to ω-automatic structures.

To the knowledge of the authors, the modulo quantifiers $\exists^{(t,k)}$ have not yet been considered for ω-automatic structures. Since an ω-automatic structure can have up to $2^{\aleph_0}$ many elements, it makes sense to consider quantifiers of the form $\exists^\varkappa$ with $\aleph_0 \le \varkappa \le 2^{\aleph_0}$.

2.2 Cardinality and modulo quantifiers for ω-automatic structures

It is the aim of this section to extend the realm of Proposition 2.2 and therefore of Theorem 2.3 to ω-automatic structures. To this aim, we fix an ω-automatic structure $\mathcal{A}$ with presentation (Γ, L, id).

Two infinite words v and w are *ultimately equal*, briefly $v \sim w$, if there exists $i \in \mathbb{N}$ such that $v(j) = w(j)$ for $j \ge i$. Since the relation $\sim$ is ω-automatic, we can assume it to be among the relations of the ω-automatic structure $\mathcal{A}$. The following lemma is our main combinatorial tool for analyzing ω-automatic structures.

Lemma 2.4. *Let M be a Büchi-automaton with n states over $\Sigma \times \Gamma$, $u \in \Sigma^\omega$, and $V = \{v \in \Gamma^\omega \mid u \otimes v \in L_\omega(M)\}$. Then $|V| = 2^{\aleph_0}$ if and only if $|V/{\sim}| > n$. Moreover, $|V| \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ and the exact number can be computed in polynomial space.*

[4] C has to be countable for otherwise the set of formulas would become uncountable rendering the decidability question nonsense.

This lemma allows to handle the quantifiers $\exists^{\aleph_0}$ and $\exists^{2^{\aleph_0}}$:

Proposition 2.5. *Let the relation $R \subseteq (\Gamma^\omega)^{n+1}$ be ω-automatic. Then the relation $R_\varkappa = \{(u_1, \dots, u_n) \mid \mathcal{A} \models \exists^\varkappa x_{n+1} : R(u_1, \dots, u_n, x_{n+1})\}$ is effectively ω-automatic for $\varkappa \in \{\aleph_0, 2^{\aleph_0}\}$.*

Proof. Let the convolution of R be accepted by an $(n+1)$-dimensional Büchi-automaton with m states. Then the formula $\exists^{2^{\aleph_0}} x_{n+1} : R(u_1, \dots, u_n, x_{n+1})$ is, by Lemma 2.4, equivalent with

$$\mathcal{A} \models \exists x_0 \cdots \exists x_m \left\{ \bigwedge_{0 \le i < j \le m} x_i \not\sim x_j \wedge \bigwedge_{0 \le i \le m} R(u_1, \dots, u_n, x_i) \right\}.$$

Lemma 2.4 also ensures that the quantifier $\exists^{\aleph_0}$ is equivalent with saying "there are infinitely, but not $2^{\aleph_0}$ many". Hence, by Proposition 2.2, $R_\varkappa$ is ω-regular. □

We now want to prove the corresponding result for modulo quantifiers. As above, let $R \subseteq (\Gamma^\omega)^{n+1}$ be ω-automatic and let $0 \le t < k \ge 2$. Because of Proposition 2.5, we can assume that for all $u_1, \dots, u_n \in \Gamma^\omega$, there are only finitely many $v \in \Gamma^\omega$ with $(u_1, \dots, u_n, v) \in R$.

For the following, it is convenient to write $\Sigma = \Gamma^n$ and consider R as an ω-automatic subset of $\Sigma^\omega \times \Gamma^\omega$. Since the convolution of R is ω-regular, it can be accepted by some *deterministic* Muller-automaton $M = (Q, \Sigma \times \Gamma, \delta, \iota, \mathcal{F})$ (see e.g. [18] for details concerning Muller automata). Now consider the alphabet $\Delta = \Sigma \times \Gamma \times \{0, \dots, k-1\}^Q \times \{0,1\}^Q$. Then one can construct a Büchi-automaton M' over Δ that accepts an ω-word $(a_i, b_i, f_i, g_i)_{i \ge 0} \in \Delta^\omega$ if and only if we have for all $i \ge 0$ and all $p \in Q$:

(1) $f_i(p) = |\{w \in \Gamma^* \mid |w| = i, \delta(\iota, a_0 a_1 \cdots a_{i-1} \otimes w) = p\}| \bmod k$ (i.e., $f(p)$ is the number of possible partners modulo k that allow $a_0 \cdots a_{i-1}$ to move from the initial state of M into p)
(2) $g_i(p) = 1$ if and only if the ω-word $a_i a_{i+1} \cdots \otimes b_i b_{i+1} \cdots$ has an accepting run in M from the state p.

To ensure condition (1), one actually constructs an automaton that counts the number of runs from ι to p whose label is of the form $a_0 a_1 \cdots a_{i-1} \otimes w$ for some word w. Since the automaton M is deterministic, this number equals the number of partners as desired.

Note that for any $u \in \Sigma^\omega$ and $v \in \Gamma^\omega$, there is precisely one ω-word $x \in L(M')$ whose projection $\pi(x)$ onto $(\Sigma \times \Gamma)^\omega$ equals $u \otimes v$.

Lemma 2.6. *Let $u \in \Sigma^\omega$ and $v \in \Gamma^\omega$, and let $x = (a_i, b_i, f_i, g_i)_{i \in \omega} \in \Delta^\omega$ be the unique ω-word with $\pi(x) = u \otimes v$. There is $i \in \mathbb{N}$ such that for all $j \ge i$, we have*

$$\sum \{f_j(p) \mid p \in Q, g_j(p) = 1\} \equiv |\{w \in \Gamma^\omega \mid w \sim v, (u, w) \in R\}| \bmod k. \quad (1)$$

Note that by our assumption on R, the set $\{w \in \Gamma^\omega \mid w \sim v, (u,w) \in R\}$ is always finite, hence the expression makes sense. The lemma thus says that the sum on the left is eventually fix and gives the number of possible parters w of u that are ultimately equal to v. From the Büchi-automaton M', we can build a new Büchi-automaton M'_s (for $0 \leq s < k$) over Δ that checks whether the sum on the left in (1) is eventually fix and equal s. Let M_s be the projection of the automaton M'_s to the alphabet $\Sigma \times \Gamma$. Then M_s accepts $u \otimes v$ if and only if, modulo k, there are s many ω-words w ultimately equal to v such that $(u,w) \in R$.

Since R is ω-automatic, there is a Büchi-automaton with, say, m states accepting the convolution of R. Let $u = (u_1, \ldots, u_n) \in \Sigma^\omega$. Since, by our assumption on R, the set $\{v \in \Gamma^\omega \mid (u,v) \in R\}$ is finite, there are $r \leq m$ many ω-words $v_1, \ldots, v_r$ in this set that are mutually not ultimately equal (Lemma 2.4). Thus, we have $\exists^{(t,k)} x_{n+1} : R(u_1, \ldots, u_n, x_{n+1})$ if and only if there exist $r \leq m$, mutually not ultimately equal words $v_1, \ldots, v_r \in \Sigma^\omega$, and integers $0 \leq t_i < k$ for $1 \leq i \leq r$ such that

1. $R(u_1, \ldots, u_n, v_i)$ for $1 \leq i \leq r$,
2. for any $v \in \Sigma^\omega$ with $R(u_1, \ldots, u_n, v)$, there exists i with $v \sim v_i$,
3. $t = \sum_{i=1}^{r} t_i \bmod k$ and $u_1 \otimes \cdots \otimes u_n \otimes v_i \in L_\omega(M_{t_i})$ for $1 \leq i \leq r$.

Since m is a constant depending on R, only, these conditions can be expressed in first-order logic. Hence Proposition 2.2 implies that $\{(u_1, \ldots, u_n) \mid \exists^{(t,k)} x_{n+1} : R(u_1, \ldots, u_n, x_{n+1})\}$ is ω-automatic. Thus, we showed:

Proposition 2.7. *Let the relation $R \subseteq (\Gamma^\omega)^{n+1}$ be ω-automatic and let $0 \leq t < k \geq 2$. Then the relation $\{(u_1, \ldots, u_n) \mid \mathcal{A} \models \exists^{(t,k)} x_{n+1} : R(u_1, \ldots, u_n, x_{n+1})\}$ is effectively ω-automatic.*

Together with Propositions 2.2 and 2.5, we obtain:

Theorem 2.8. *Let $\mathcal{A}$ be an ω-automatic structure and let C be an at most countably infinite set of cardinals. Then the* $\mathrm{FO}(\exists^\infty, (\exists^\varkappa)_{\varkappa \in C}, (\exists^{(t,k)})_{0 \leq t < k > 1})$*-theory of $\mathcal{A}$ is decidable.*

Proof. Lemma 2.4 implies that a formula of the form $\exists^\varkappa x\, \psi$ with $\aleph_0 < \varkappa < 2^{\aleph_0}$ can never be true in $\mathcal{A}$. Hence, the theory in question can be reduced to the $\mathrm{FO}(\exists^\infty, \exists^{\aleph_0}, \exists^{2^{\aleph_0}}, (\exists^{(t,k)})_{0 \leq t < k > 1})$-theory of $\mathcal{A}$. Since emptiness of Büchi-automata is decidable, the result follows from Propositions 2.2, 2.5, and 2.7. ☐

3 ω-automatic structures of bounded degree and complexity of theories

As first observed in [4], there are automatic structures with a non-elementary first-order theory. Our aim in this section is to single out a class of ω-automatic structures such that the $\mathrm{FO}(\exists^\infty, \exists^{\aleph_0}, \exists^{2^{\aleph_0}}, (\exists^{(t,k)})_{0 \leq t < k > 1})$-theory is elementarily decidable. In doing so, we will find that even more general quantifiers give rise to elementarily decidable theories provided we constrain ourselves to structures of bounded degree.

3.1 Definitions and known results

Structures of bounded degree. Let $\mathcal{A}$ be a τ-structure with universe A. The *Gaifman-graph* $G_{\mathcal{A}}$ of the structure $\mathcal{A}$ is the following undirected graph:

$$G_{\mathcal{A}} = (A, \{(a,b) \in A \times A \mid \exists R \in \tau\, \exists (c_1, \ldots, c_{n_R}) \in R\, \exists j,k : c_j = a \neq b = c_k\}).$$

Thus, the set of nodes is the universe of $\mathcal{A}$ and there is an edge between two elements, if and only if they are contained in some tuple belonging to one of the relations of $\mathcal{A}$. The structure $\mathcal{A}$ is *locally finite*, if every node of the Gaifman-graph $G_{\mathcal{A}}$ has only finitely many neighbors. It has *bounded degree*, if its Gaifman-graph $G_{\mathcal{A}}$ has bounded degree, i.e., there exists a constant d such that every $a \in A$ is adjacent to at most d other nodes in $G_{\mathcal{A}}$.

In contrast to the general case, if the degree of the automatic structure $\mathcal{A}$ is bounded, an elementary upper bound for the first-order theory of $\mathcal{A}$ is due to the second author (we define $\exp(1,n) = 2^n$ and $\exp(k+1,n) = 2^{\exp(k,n)}$):

Theorem 3.1 ([16]). *If $\mathcal{A}$ is an automatic structure of bounded degree, then the* FO*-theory of $\mathcal{A}$ can be decided in* $\mathrm{SPACE}(\exp(3, O(n)))$ *and there is such a structure for which* $\mathrm{SPACE}(\exp(2, O(n)))$ *is a lower bound.*

This result was not known to apply to more general quantifiers nor to ω-automatic structures. An important tool in the proof of Theorem 3.1 as well as in our extension, is the concept of a sphere that we introduce next.

With $d_{\mathcal{A}}(a,b)$, where $a, b \in A$, we denote the distance between a and b in $G_{\mathcal{A}}$, i.e., it is the length of a shortest path connecting a and b in $G_{\mathcal{A}}$. For $a \in A$ and $r \geq 0$ we denote with $S_{\mathcal{A}}(r,a) = \{b \in A \mid d_{\mathcal{A}}(a,b) \leq r\}$ the *r-sphere around a*. If $\bar{a} = (a_1, \ldots, a_n) \in A^n$ is a tuple, then $S_{\mathcal{A}}(r, \bar{a}) = \bigcup_{i=1}^{n} S_{\mathcal{A}}(r, a_i)$. The *neighborhood* $N_{\mathcal{A}}(r,\bar{a}) = \mathcal{A}{\restriction} S_{\mathcal{A}}(r,\bar{a})$ of radius r around $\bar{a}$ is the substructure of $\mathcal{A}$ induced by $S_{\mathcal{A}}(r,\bar{a})$.

Generalized quantifiers and locality. Let us fix a relational signature τ. In this section, we will consider the logic $\mathcal{L}(\mathcal{Q}_u)$. Formulas of the logic $\mathcal{L}(\mathcal{Q}_u)$ are built from atomic formulas of the form $R(x_1, \ldots, x_{n_R})$, where $R \in \tau$ is a relational symbol and $x_1, \ldots, x_{n_R}$ are first-order variables ranging over the universe of the underlying structure, using boolean connectives and quantifications of the form $\mathcal{Q}_{\mathcal{C}} y : (\psi_1(\bar{x}, y), \ldots, \psi_n(\bar{x}, y))$. Here, $\psi_i(\bar{x}, y)$ is already a formula of $\mathcal{L}(\mathcal{Q}_u)$, $\bar{x}$ is a sequence of variables, and $\mathcal{C}$ is an n-ary relation over cardinals, i.e., $\mathcal{C} = \{(\varkappa_{i,1}, \ldots, \varkappa_{i,n}) \mid i \in J, \varkappa_{i,j} \text{ is a cardinal}\}$ for some index set J. To define the semantics of the $\mathcal{Q}_{\mathcal{C}}$-quantifier, let $\mathcal{A}$ be a τ-structure with universe A and let $\bar{u}$ be a tuple of values from A of the same length as $\bar{x}$. Then $\mathcal{A} \models \mathcal{Q}_{\mathcal{C}} y : (\psi_1(\bar{u}, y), \ldots, \psi_n(\bar{u}, y))$ if and only if $(\varkappa_1, \ldots, \varkappa_n) \in \mathcal{C}$, where $\varkappa_i$ is the cardinality of the set $\{a \in A \mid \mathcal{A} \models \psi_i(\bar{u}, a)\}$. In the above situation, we call the quantifier $\mathcal{Q}_{\mathcal{C}}$ also an *n-dimensional counting quantifier*. The *quantifier rank* $\mathrm{qfr}(\varphi)$ of a formula φ is the maximal number of nested quantifiers of φ. The logic $\mathcal{L}(\mathcal{Q}_u)$ is a finitary fragment of the logic $\mathcal{L}_{\infty,\omega}(\mathcal{Q}_u)^{\omega}$ from [10], which allows infinite conjunctions and disjunctions but restricts to finite quantifier rank.

Let us consider some examples for generalized quantifiers. The ordinary existential quantifier $\exists y : \varphi(\bar{x}, y)$ is equivalent to $\mathcal{Q}_{\mathcal{C}}\, y : \varphi(\bar{x}, y)$, where $\mathcal{C}$ is the class of all non-zero cardinals. Similarly, we can obtain the counting quantifier $C_K\, y : \varphi(\bar{x}, y)$ for K some class of cardinals ("the number of y satisfying $\varphi(\bar{x}, y)$ belongs to K"). Well-known special cases of the latter quantifier are the quantifiers $\exists^\infty$, $\exists^\varkappa$, and $\exists^{(t,q)}$ from the Section 2. All these counting quantifiers are one-dimensional. A well-known two-dimensional counting quantifier is the Härtig quantifier $I\, y : (\psi_1(\bar{x}, y), \psi_2(\bar{x}, y))$ ("the number of y satisfying $\psi_1(\bar{x}, y)$ equals the number of y satisfying $\psi_2(\bar{x}, y)$"). For this we have to choose for $\mathcal{C}$ the identity relation on cardinals.

For a class $\mathbb{C}$, where every $\mathcal{C} \in \mathbb{C}$ is a relation on cardinals, $\mathrm{FO}(\mathbb{C})$ denotes those formulas of $\mathcal{L}(\mathcal{Q}_u)$ that only use quantifiers of the form $\mathcal{Q}_{\mathcal{C}}$ with $\mathcal{C} \in \mathbb{C}$ along with the existential quantifier $\exists$. For a singleton class $\mathbb{C} = \{\mathcal{C}\}$ we also write $\mathrm{FO}(\mathcal{C})$ instead of $\mathrm{FO}(\mathbb{C})$.

We will make use of the following locality principle for the logic $\mathcal{L}(\mathcal{Q}_u)$:

Theorem 3.2 ([10]). *Let $\mathcal{A}$ be a locally finite structure, let $\varphi(x_1, \ldots, x_k)$ be an $\mathcal{L}(\mathcal{Q}_u)$-formula of quantifier rank at most d, and let $\bar{a}, \bar{b} \in \mathcal{A}^k$ be k-tuples with $(N_{\mathcal{A}}(2^d, \bar{a}), \bar{a}) \cong (N_{\mathcal{A}}(2^d, \bar{b}), \bar{b})$.[5] Then $\mathcal{A} \models \varphi(\bar{a})$ if and only if $\mathcal{A} \models \varphi(\bar{b})$.*

Proof. Keisler and Lotfallah [10] proved this statement for locally finite *countable* structures. As an intermediate step, they considered an infinitary logic with counting quantifiers C_A with $A = \{0, 1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. Considering, instead, counting quantifiers C_A with $A = \{\lambda \mid \lambda \leq \varkappa\}$ for $\varkappa$ a cardinal, one obtains the above general theorem (which does not restrict to countable structures) without any further modifications of [10]. □

3.2 Complexity of the $\mathcal{L}(Q_u)$-theory

In Section 3.4 we will show that there exists a locally finite automatic structure $\mathcal{A}$ and a recursive set $K \subseteq \mathbb{N}$ such that the $\mathrm{FO}(C_K)$-theory of $\mathcal{A}$ is undecidable. To obtain a decidability result, we therefore consider an ω-automatic structure $\mathcal{A}$ of bounded degree. We will consider the $\mathrm{FO}(\mathbb{C})$-theory of $\mathcal{A}$, where every $\mathcal{C} \in \mathbb{C}$ is a relation over cardinals. Furthermore, we make the following assumptions:

(1) (Γ, L, id) is an ω-automatic presentation for $\mathcal{A}$, i.e., in particular L is the universe of $\mathcal{A}$.
(2) $\delta \in \mathbb{N}$ is a bound for the degrees of the nodes in the Gaifman graph $G_{\mathcal{A}}$.
(3) For every $0 \leq n \leq \delta$ the signature τ contains a unary predicate $\deg_n$ with $\mathcal{A} \models \deg_n(u)$ if and only if the degree of u in the Gaifman-graph $G_{\mathcal{A}}$ is exactly n.
(4) $\mathbb{C}$ is a countable set of relations on $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

[5] This means that there exists an isomorphism $f : N_{\mathcal{A}}(2^d, \bar{a}) \to N_{\mathcal{A}}(2^d, \bar{b})$ mapping for every $1 \leq i \leq k$ the i-th entry of $\bar{a}$ to the i-th entry of $\bar{b}$.

Clearly, neither (1) nor (2) imposes restrictions on (the isomorphism type of) $\mathcal{A}$. Since the set of nodes w of degree n is first-order definable, it is ω-regular. Hence we can assume it to be among the relations of $\mathcal{A}$. Thus, (3) is no essential restriction. Finally, consider (4). If $\mathbb{C}$ allows more than countably many relations, then it does not make sense to ask for the decidability of the FO($\mathbb{C}$)-theory of $\mathcal{A}$ since it is uncountable. Furthermore, one can show that even without restricting to relations over $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$, the size of any definable set belongs to $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. Hence we can safely assume (4).

We will prove that under the above four restrictions, the FO($\mathbb{C}$)-theory of $\mathcal{A}$ can be reduced in triply exponential space to the relations in $\mathbb{C}$. For this, we need the following concept: A pair $(\mathcal{B}, \bar{b})$ is a *potential* (D,k)*-sphere* (for $D, k \in \mathbb{N}$) if the following holds:

- $\mathcal{B}$ is a finite τ-structure whose Gaifman-graph has degree at most δ,
- $\bar{b}$ is a k-tuple of elements from $\mathcal{B}$,
- $N_{\mathcal{B}}(2^D, \bar{b}) = \mathcal{B}$, i.e., every element of $\mathcal{B}$ has distance at most 2^D from some entry of the tuple $\bar{b}$,
- for any $y \in S_{\mathcal{B}}(2^D - 1, \bar{b})$, we have $\mathcal{B} \models \deg_n(y)$ if and only if n is the degree of y in the Gaifman-graph of $\mathcal{B}$, and
- for any $y \in \mathcal{B} \setminus S_{\mathcal{B}}(2^D - 1, \bar{b})$ there is a unique $0 \leq n \leq \delta$ such that $\mathcal{B} \models \deg_n(y)$ and the degree of y in the Gaifman-graph of $\mathcal{B}$ is at most n.

Thus, a potential (D,k)-sphere is a candidate for a 2^D-sphere around some k-tuple in the structure $\mathcal{A}$.

Let $\{b_1, b_2, \ldots, b_n\}$ be the universe of $\mathcal{B}$ with $\bar{b} = (b_1, \ldots, b_k)$ $(k \leq n)$. Since $\bar{b}$ is not necessarily repetition-free, we may have $b_i = b_j$ for $i \neq j$ in case $i, j \leq k$, but we may assume that $b_{k+1}, \ldots, b_n$ are pairwise different and different from $b_1, \ldots, b_k$. We define $\varphi_{(\mathcal{B},\bar{b})}(x_1, \ldots, x_k) = \exists x_{k+1} \cdots \exists x_n : \psi(x_1, \ldots, x_n)$, where $\psi(x_1, \ldots, x_n)$ is the conjunction of the following formulas:

- $x_i = x_j$ if $b_i = b_j$ and $x_i \neq x_j$ if $b_i \neq b_j$
- $R(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ if $(b_{i_1}, b_{i_2}, \ldots, b_{i_m}) \in R$ for $R \in \tau$ with $m = n_R$ and $i_1, \ldots, i_m \in \{1, \ldots, n\}$
- $\neg R(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$ if $(b_{i_1}, b_{i_2}, \ldots, b_{i_m}) \notin R$ for $R \in \tau$ with $m = n_R$ and $i_1, \ldots, i_m \in \{1, \ldots, n\}$.

Lemma 3.3. *There exists a constant $c \in \mathbb{N}$ such that for any potential (D,k)-sphere $(\mathcal{B}, \bar{b})$, the existential FO-formula $\varphi_{(\mathcal{B},\bar{b})}$ has size at most* $\exp(2, c(D+k))$. *For any k-tuple $\bar{u} \in L^k$, we have: $\mathcal{A} \models \varphi_{(\mathcal{B},\bar{b})}(\bar{u}) \Leftrightarrow (N_{\mathcal{A}}(2^D, \bar{u}), \bar{u}) \cong (\mathcal{B}, \bar{b})$.*

Lemma 3.4. *There are functions $\# : \mathbb{N}^2 \to \mathbb{N}$ and $\Phi : \mathbb{N}^3 \to \mathrm{FO}$ such that*

1. *$\#(D,k)$ is computable in space $\exp(2, O(D+k))$ and $\Phi(D,k,i)$ in space $\exp(2, O(D+k)) + \log(i)$*
2. *for any $D, k \in \mathbb{N}$, $\#(D,k)$ is the number of potential (D,k)-spheres,*
3. *for any $D, k, i \in \mathbb{N}$, there exists a potential (D,k)-sphere $\mathcal{B}(D,k,i)$ with $\varphi_{\mathcal{B}(D,k,i)} = \Phi(D,k,i)$, and*

4. for any $D, k \in \mathbb{N}$ and any potential (D,k)-sphere $(\mathcal{B}, \bar{b})$, there exists $1 \leq i \leq \#(D,k)$ with $\varphi_{(\mathcal{B},\bar{b})} = \Phi(D,k,i)$.

Note that $\mathcal{B}(D,k,1), \ldots, \mathcal{B}(D,k,\#(D,k))$ enumerates the isomorphism types of potential (D,k)-spheres for any $D, k \in \mathbb{N}$.

In the following we identify a tuple $\bar{u} = (u_1, \ldots, u_k)$ with its convolution $u_1 \otimes u_2 \otimes \cdots \otimes u_k$. We write $k = |\bar{u}|$ for the length of the tuple $\bar{u}$.

Lemma 3.5. *The following can be computed in space* $\exp(3, O(D+k)) + \log(i)$*:*
INPUT: $D, k, i \in \mathbb{N}$
QUTPUT: a k-dimensional Büchi-automaton M of size $\exp(3, O(D+k))$ with
$L_\omega(M) = \{\bar{u} \mid (N_{\mathcal{A}}(2^D, \bar{u}), \bar{u}) \cong \mathcal{B}(D,k,i)\}$.

Let us fix a function $s(D+k) \in \exp(3, O(D+k))$ bounding the space in Lemma 3.5. For a word $u \in \Sigma^\omega$, its *norm* $\lambda(u)$ is $\lambda(u) = \inf\{|vw| \mid u = vw^\omega\}$, with $\lambda(u) = \infty$ if u is not ultimately periodic, i.e., not of the form vw^ω for some $v, w \in \Sigma^*$. Let UP denote the class of all ultimately periodic ω-words over some alphabet. In the algorithms below, we will often handle ω-words $u \in \mathrm{UP}$ that can be given as a pair (v, w) with $u = vw^\omega$ and $|vw| = \lambda(w)$. Note that if M is a Büchi-automaton with n states and $L_\omega(M) \neq \emptyset$, then we find an ω-word $u \in L_\omega(M)$ such that $\lambda(u) \leq 2n$. Note that for $\bar{u} = (u_1, \ldots, u_k)$ we have $\lambda(\bar{u}) = \lambda(u_1 \otimes u_2 \cdots \otimes u_k) \leq \prod_{1 \leq i \leq k} \lambda(u_i)$. Since we can build a $(k+1)$-dimensional Büchi-automaton with $\lambda(\bar{u})$ many states that accepts the language $\bar{u} \otimes \Sigma^\omega$, the product construction for Büchi-automata and Lemma 3.5 gives:

Lemma 3.6. *The following can be computed in space $3 \cdot s(D+k+1) \cdot \lambda(\bar{u}) + \log(i)$ if $k = |\bar{u}| > 0$ and in space $s(D+1) + \log(i)$ if $k = |\bar{u}| = 0$:*
INPUT: $D, k, i \in \mathbb{N}$ and $\bar{u} \in L^k \cap \mathrm{UP}$
OUTPUT: a $(k+1)$-dimensional Büchi-automaton M with $L_\omega(M) = \{\bar{u}w \in L^{k+1} \mid (N_{\mathcal{A}}(2^D, \bar{u}w), \bar{u}w) \cong \mathcal{B}(D,k+1,i)\}$.
Moreover, if $L_\omega(M) \neq \emptyset$, then we can compute within the same space bound a word $w \in L \cap \mathrm{UP}$ with $\bar{u}w \in L_\omega(M)$ and

$$\lambda(w) \leq \begin{cases} 6 \cdot s(D+k+1) \cdot \lambda(\bar{u}) & \text{if } k > 0 \\ 2 \cdot s(D+1) & \text{if } k = 0. \end{cases} \qquad (*)$$

Now consider the following two algorithms **size** and **check**. The algorithm **size** shall return the number of words $v \in \Sigma^\omega$ with $\mathcal{A} \models \varphi(\bar{u}v)$. The algorithm **check** shall check whether $\mathcal{A} \models \varphi(\bar{u})$.

check$(\varphi(\bar{x}), \bar{u}) : \{0, 1\}$
    ($\varphi(\bar{x})$ formula with $|\bar{u}| = |\bar{x}|$ many free variables,
    $\bar{u}$ tuple of ultimately periodic words from L)
  case $\varphi = R(\bar{x})$
    if $\bar{u} \in R$ **then return**(1) **else return**(0) **endif**
  case $\varphi = \varphi_1 \wedge \varphi_2$
    return(**check**$(\varphi_1, \bar{u}) \wedge$ **check**$(\varphi_2, \bar{u})$)

8 **case** $\varphi = \neg\varphi_1$
9 **return**($\neg$**check**$(\varphi_1, \bar{u})$)
10 **case** $\varphi = \mathcal{Q}_{\mathcal{C}} y : (\psi_1(\bar{x}, y), \ldots, \psi_n(\bar{x}, y))$
11 **for** $i = 1$ **to** n **do**
12 $\varkappa_i :=$ **size**$(\psi_i, \bar{u})$
13 **endfor**
14 **if** $(\varkappa_1, \ldots, \varkappa_n) \in \mathcal{C}$ **then return**(1) **else return**(0) **endif**

1 **size**$(\varphi, \bar{u}) : \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$
2 (φ formula with $|\bar{u}| + 1$ many free variables,
3 $\bar{u}$ tuple of ultimately periodic words from L)
4 $D := \mathrm{qfr}(\varphi)$; $\varkappa := 0$;
5 **for** $i := 1$ **to** $\#(D, |\bar{u}| + 1)$ **do**
6 **calculate** an $|\bar{u}| + 1$-dimensional Büchi-automaton M with
$L_\omega(M) = \{\bar{u}w \in L^{|\bar{u}|+1} \mid (N_{\mathcal{A}}(2^D, \bar{u}w), \bar{u}w) \cong \mathcal{B}(D, |\bar{u}| + 1, i)\}$
7 **if** $L_\omega(M) \neq \emptyset$ **then**
8 **choose** $w \in \Sigma^\omega$ with $\bar{u}w \in L_\omega(M)$ and $\lambda(w) \leq 2 \cdot s(D + 1)$
9 if $|\bar{u}| = 0$ and $\lambda(w) \leq 6 \cdot s(D + |\bar{u}| + 1) \cdot \lambda(\bar{u})$ otherwise
10 **if check**$(\varphi, \bar{u}w)$ **then**
11 $\varkappa := \varkappa + |L_\omega(M)|$
12 **endif**
13 **endif**
14 **endfor**
15 **return**$(\varkappa)$

Let us first verify the correctness of these algorithms. If **size** behaves as intended, the correctness of **check** is rather obvious. We now discuss **size**. By Lemma 3.4, line 5 iterates over all potential $(D, |\bar{u}| + 1)$-spheres. Since $D = \mathrm{qfr}(\varphi)$, there *exists* a tuple $\bar{u}w \in L_\omega(M)$ with $\mathcal{A} \models \varphi(\bar{u}w)$ if and only if $\mathcal{A} \models \varphi(\bar{u}v)$ for *all* $\bar{u}v \in L_\omega(M)$ by Theorem 3.2, where M is the Büchi-automaton calculated in line 6. Therefore, we select in line 8,9 a "short" tuple $\bar{u}w \in L_\omega(M)$ and check in line 10 whether $\mathcal{A} \models \varphi(\bar{u}w)$ using algorithm **check**. If this is true, then we add to the current $\varkappa$ the size of the language $L_\omega(M)$, which can be calculated by Lemma 2.4 in polynomial space wrt. the size of M.

Next we discuss the space complexity of a call **check**(ψ, ε) (where ε is the empty tuple) for a sentence ψ of quantifier rank D_0. Note that when we call **size** with parameters φ and $\bar{u}$, then $\mathrm{qfr}(\varphi) + |\bar{u}| + 1 \leq D_0$. Thus, the Büchi-automaton M in line 6 can be calculated in space $3 \cdot s(D + |\bar{u}| + 1) \cdot \lambda(\bar{u}) \leq 3 \cdot s(D_0) \cdot \lambda(\bar{u})$ by Lemma 3.6 (since $i \leq \#(D, |\bar{u}| + 1) \in \exp(3, O(D_0))$, we can forget the summand $\log(i)$) and also the bound $2 \cdot s(D + 1) \leq 2 \cdot s(D_0)$ (resp. $6 \cdot s(D + |\bar{u}| + 1) \cdot \lambda(\bar{u}) \leq 6 \cdot s(D_0) \cdot \lambda(\bar{u})$) in line 8,9 for the ω-word w follows from Lemma 3.6. Assume that $(u_1, u_2, \ldots, u_{D_0})$ is the tuple of ultimately periodic ω-words calculated by the algorithm. If we set $\bar{u}_k = (u_1, u_2, \ldots, u_k)$, then we obtain:

$$\lambda(\bar{u}_1) \leq 2 \cdot s(D_0) \qquad \text{(by } (*) \text{ in Lemma 3.6)}$$
$$\lambda(\bar{u}_{k+1}) \leq \lambda(\bar{u}_k) \cdot \lambda(u_{k+1}) \leq 6 \cdot s(D_0) \cdot \lambda(\bar{u}_k)^2$$

From this, we obtain by induction $\lambda(\bar{u}_k) \leq 2^{2^k} \cdot 6^{2^k-1} \cdot s(D_0)^{2^k-1}$. Since $s(D_0) \in \exp(3, O(D_0))$ and $k \leq D_0$, it follows $\lambda(\bar{u}_k) \in \exp(3, O(D_0))$. Hence, each of the Büchi-automata M in line 6 can be constructed in triply-exponential space. Since the recursion depth of the overall algorithm is bounded by the size of the input formula and for each recursive call only a triply exponential amount of information has to be stored, the whole algorithm can be executed in space triply exponential in the size of the input formula. Thus, we proved:

Theorem 3.7. *Let $\mathbb{C} = \{\mathcal{C}_i \mid i \in \mathbb{N}\}$ be a countable set of relations on $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. Let $\mathcal{A}$ be an ω-automatic structure of bounded degree. Then the* FO($\mathbb{C}$)*-theory of $\mathcal{A}$ can be decided in triply exponential space by a Turing machine with oracle $\{(i, \bar{c}) \mid i \in \mathbb{N}, \bar{c} \in \mathcal{C}_i\}$.*

3.3 Expressiveness of the logic $\mathcal{L}(Q_u)$

Let $\mathcal{A}$ be some structure of bounded degree and let $\varphi(\bar{x})$ be an $\mathcal{L}(Q_u)$-formula with k free variables of quantifier depth d. We want to show that there exists an equivalent first-order formula $\psi(\bar{x})$. For this, we can first extend the signature of $\mathcal{A}$ by the first-order definable relations $\deg_n$ in order to ensure assumptions (2) and (3) from page 9. Now let $\#$ and Φ be the functions from Lemma 3.4 and set

$$I = \{i \mid 1 \leq i \leq \#(d, k), \mathcal{A} \models \forall \bar{x} : (\Phi(d, k, i) \to \varphi)\}$$

and $\psi = \bigvee_{i \in I} \Phi(d, k, i)$. Then Lemmas 3.3 and 3.4 together with Theorem 3.2 imply $\mathcal{A} \models \forall \bar{x}(\varphi \leftrightarrow \psi)$. This proves:

Corollary 3.8. *Let $\mathcal{A}$ be a τ-structure of bounded degree, and let $\varphi(\bar{x}) \in \mathcal{L}(\mathcal{Q}_u)$. There exists a formula $\psi(\bar{x}) \in$ FO such that $\mathcal{A} \models \forall \bar{x}(\varphi \leftrightarrow \psi)$.*

The above proof is not effective since it does not give a way to compute the set I effectively. For ω-automatic structures $\mathcal{A}$ of bounded degree, the situation changes since it can be decided in elementary space as to whether $\alpha_i = \forall \bar{x}(\Phi(d, k, i) \to \varphi)$ holds in $\mathcal{A}$:

Corollary 3.9. *Let $\mathbb{C} = \{\mathcal{C}_i \mid i \in \mathbb{N}\}$ be a countable set of relations on $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. Let $\mathcal{A}$ be an ω-automatic structure of bounded degree. For any $\varphi(\bar{x}) \in$* FO($\mathbb{C}$)*, one can construct in elementary space (modulo $\mathbb{C}$) a formula $\psi(\bar{x}) \in$* FO *and a $|\bar{x}|$-dimensional Büchi-automaton M such that for any $\bar{u} \in L^{|\bar{x}|}$:*

$$\mathcal{A} \models \varphi(\bar{u}) \iff \mathcal{A} \models \psi(\bar{u}) \iff \bar{u} \in L_\omega(M) \ .$$

Recall that by Propositions 2.2, 2.5, and 2.7, any relation definable in FO extended by modulo- and cardinality-quantifiers is effectively ω-automatic. A similar statement can be found in Corollary 3.9. Also Theorems 2.8 and 3.7 are similar in as far as they state the decidability of some theories. But the proof strategies are different: while Theorem 2.8 was derived from Propositions 2.2, 2.5, and 2.7, the corresponding statement Theorem 3.7 was used to prove Corollary 3.9.

3.4 Optimality

The main results concerning the powerful logic $\mathcal{L}(Q_u)$ deal with structures satisfying two assumptions: they are ω-automatic and of bounded degree. In this section, we show that the two assumptions we made cannot be relaxed. First, it is shown that relaxing "ω-automatic" to "recursive" makes the results fail:

Theorem 3.10. *There exists a recursive structure $\mathcal{A}$ of bounded degree such that the* FO*-theory of $\mathcal{A}$ is decidable and the* FO$(\exists^\infty)$*-theory of $\mathcal{A}$ is undecidable.*

Proof. Let $L \subseteq \{0,1\}^*$ be a recursively enumerable, but not recursive set and let M be a Turing machine that, on input of $w \in \{0,1\}^*$, eventually stops if and only if $w \in L$. Let $f(w) \in \mathbb{N} \cup \{\omega\}$ denote the number of steps M performs on input w. The structure $\mathcal{A}$ consists of $f(w)$ many copies of the word $\rhd w \lhd$ for any $w \in \{0,1\}^*$ (seen as labeled finite successor structures), i.e., $\mathcal{A}$ is a labeled directed graph whose degree is bounded by 2. Then in FO$(\exists^\infty)$, we can express that M does not stop on input w, hence this theory is undecidable. Gaifman's theorem, on the other hand, yields that the FO-theory is decidable. $\square$

By choosing a more complicated but still recursive counting quantifier, we can show that Theorem 3.7 even fails for locally finite automatic structures.

Theorem 3.11. *There is a recursive set $K \subseteq \mathbb{N}$ and a locally finite automatic structure $\mathcal{A}$ such that the* FO(C_K)*-theory of $\mathcal{A}$ is undecidable.*

Proof. We start with the structure $(\mathbb{N}, +1)$ and attach, to any element $n \in \mathbb{N}$, additional n nodes via a relation t. The resulting structure $\mathcal{A}$ is automatic. Let $a_1, a_2, a_3, \dots$ be a recursive enumeration of the non-recursive set $A \subseteq \mathbb{N}$ and let K denote the recursive set $\{a_1 + \cdots + a_i \mid i \geq 1\}$. Let $\varphi_K(x)$ be the formula $C_K y : t(x,y)$. Then $m \in A$ if and only if there exists y satisfying $\varphi_K(y) \wedge \varphi_K(y+m) \wedge \bigwedge_{1 \leq k < m} \neg\varphi_K(y+k)$. $\square$

4 An open problem

In view of Theorems 2.8 and 3.11 it might be an interesting problem to characterize those subsets $K \subseteq \mathbb{N}$ such that for every (ω-)automatic structure (not necessarily of bounded degree), the FO(C_K)-theory of $\mathcal{A}$ is decidable. Note that by Theorem 2.8, this is true for every semi-linear set K. Since $(\mathbb{N}, \leq)$ is automatic and since $x \in K$ can be expressed as $C_K y : y < x$, the set K has to be decidable. As observed by an of the referees, even the monadic second order theory of $(\mathbb{N}, \leq, K)$ has to be decidable since it can be reduced to the FO(C_K)-theory of the ω-automatic structure $(\{0,1\}^\infty, \leq)$. Furthermore, K cannot be the range of any non-linear polynomial over $\mathbb{N}$ [5] nor can it be k-recognizable (for some k) but not semi-linear [2].

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