The Isomorphism Problem On Classes of Automatic Structures with Transitive Relations

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Abstract. Automatic structures are finitely presented structures where the universe and all relations can be recognized by finite automata. It is known that the isomorphism problem for automatic structures is complete for Σ^1_1, the first existential level of the analytical hierarchy. Positive results on ordinals and on Boolean algebras raised hope that the isomorphism problem is simpler for transitive relations. We prove that this is not the case. More precisely, this paper shows:

(i) The isomorphism problem for automatic equivalence relations is complete for Π^0_1 (first universal level of the arithmetical hierarchy).
(ii) The isomorphism problem for automatic trees of height $n \geq 2$ is Π^0_{2n-3}-complete.
(iii) The isomorphism problem for well-founded automatic order trees is recursively equivalent to true first-order arithmetic.
(iv) The isomorphism problem for automatic order trees is Σ^1_1-complete.
(v) The isomorphism problem for automatic linear orders is Σ^1_1-complete.

We also obtain Π^0_1-completeness of the elementary equivalence problem for several classes of automatic structures and Σ^1_1-completeness of the isomorphism problem for linear orders consisting of a deterministic context-free language together with the lexicographic order. This solves several open questions of Ésik, Khoussainov, Nerode, Rubin, and Stephan.

1 Introduction

The idea of an automatic structure goes back to Büchi and Elgot who used finite automata to decide, e.g., Presburger arithmetic [9]. Automaton decidable theories [16] and automatic groups [11] are similar concepts. A systematic study was initiated by Khoussainov and Nerode [21] who also coined the name "*automatic structure*". In essence, a structure is automatic if the elements of the universe can be encoded by strings from a regular language and every relation of the structure can be recognized by a finite state automaton with several heads that proceed synchronously. Automatic structures received increasing interest over the last years [1,4,19,20,22,23,24,26,27,33]. One of the main motivations for investigating automatic structures is that their first-order theories can be decided uniformly (i.e., the input is an automatic presentation and a first-order sentence).

Automatic structures form a subclass of computable (or recursive) structures. A structure is computable, if its domain as well as all relations are computable sets of finite words (or naturals). A typical example of a computable structure is $(\mathbb{N}; +, \times)$. Computable structures are the starting point of the rich field of computable model theory [12]. A well-studied problem in this field is the isomorphism problem, which asks whether two given computable structures over the same signature (encoded by Turing-machines for the domain and all relations) are isomorphic. It is well known that the isomorphism problem for computable structures is complete for the first level of the analytical hierarchy Σ^1_1. In fact, Σ^1_1-completeness holds for many subclasses of computable structures, e.g., for linear orders, trees, undirected graphs, Boolean algebras, Abelian p-groups, see [5,13]. These papers also show, as a consequence of the Σ^1_1-completeness of the isomorphism problem, that these classes do not have a "good classification" (more precisely: they do not have a hyperarithmetical Friedberg enumeration).

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In [23], it was shown that also for automatic structures the isomorphism problem is Σ^1_1-complete. By a direct interpretation, it follows that for the following classes the isomorphism problem is still Σ^1_1-complete [29]: automatic successor trees, automatic undirected graphs, automatic commutative monoids, automatic partial orders (of height 2), automatic lattices (of height 4), and automatic 1-ary functions. On the other hand (and in contrast to computable structures), the isomorphism problem is decidable for automatic ordinals [24] and automatic Boolean algebras [23]. An intermediate class is the class of all locally-finite automatic graphs, for which the isomorphism problem is complete for Π^0_3 (third level of the arithmetical hierarchy) [32].

Although the interpretation technique from [29] yields undecidability for the isomorphism problem for automatic partial orders, it seems to be difficult to extend this technique to more restricted classes like order trees (i.e., trees seen as particular partial orders) or linear orders. Moreover, the automatic graphs constructed in [23] are based on transition graphs of Turing-machines. But the transitive closure of such a transition graph is in general not automatic (its first-order theory is undecidable in general). Hence, the techniques from [23,29] do not work for automatic order trees, automatic linear orders, and also automatic equivalence relations. Moreover, as mentioned above, for some very restricted classes of automatic structures (ordinals [24], Boolean algebras [23]), the isomorphism problem is decidable. Recent surveys consequently ask whether the isomorphism problem is decidable for automatic equivalence relations and automatic linear orders [22,33]. For automatic equivalence relations, it is conjectured in [22] that the isomorphism problem is decidable. For automatic linear orders, already [24] asked for the decidability status of the isomorphism problem.

Contrary to the hopes expressed in [22,24,33], our main results are:

- The isomorphism problem for automatic equivalence relations is Π^0_1-complete.
- The isomorphism problem for automatic trees of finite height $n \geq 2$ (where the height of a tree is the maximal number of edges along a path) is Π^0_{2n-3}-complete.
- The isomorphism problem for automatic well-founded order trees is recursively equivalent to true arithmetic (the first-order theory of $(\mathbb{N}; +, \times)$).
- The isomorphism problem for automatic order trees is Σ^1_1-complete.
- The isomorphism problem for automatic linear orders is Σ^1_1-complete.

Contrary to the above-mentioned technique using configuration graphs of Turing machines, our proofs are based on the undecidability of Hilbert's 10^{th} problem. Recall that Matiyasevich proved that every recursively enumerable set of natural numbers is Diophantine, i.e., the projection to the first component of the set of zeros of some polynomial $p(\overline{x}) \in \mathbb{Z}[\overline{x}]$ [28]. Honkala inferred that it is undecidable whether the range of a rational power series equals $\mathbb{N}$ [17]. His basic idea was to construct from a given polynomial $p(\overline{x})$ in several variables a finite automaton, such that on a word encoding an input tuple $\overline{c}$ for the polynomial, the automaton has exactly $p(\overline{c})$ many accepting runs, see also [34, Theorem II.11.1]. Using a similar encoding, we show that the isomorphism problem for automatic equivalence relations is Π^0_1-complete, which answers Question 4.2 from [22] negatively. By the same arguments, we obtain that elementary equivalence of automatic equivalence relations (i.e., the problem whether two equivalence relations have the same first-order theory) is Π^0_1-complete as well, thereby partially answering Question 4.8 from [22].

Next, we extend our technique in order to show that the isomorphism problem for automatic successor trees of height $n \geq 2$ is Π^0_{2n-3}-complete. Using the easy correspondence between equivalence structures and trees of height 2, our result for equivalence relations makes up, in some sense, the induction base $n = 2$. Our result for automatic trees of finite height gives an answer to Question 4.6 from [22], which asks for natural classes of automatic structures, for which the isomorphism problem is complete for levels of the arithmetical hierarchy.

For arbitrary automatic trees, we prove that the isomorphism problem has maximal complexity, i.e., that it is Σ^1_1-complete. For automatic successor trees, Σ^1_1-completeness of the isomorphism problem was already shown in [23]. Here, the crucial point is that we consider order trees (for trees of finite height, the distinction between order trees and successor trees is not important). Our proof for Σ^1_1-completeness on automatic order trees is based on a reduction from the isomorphism

problem for computable trees. To achieve this reduction, we combine our techniques for automatic trees of finite height with the method from [23].

Finally, using a similar but technically more involved reduction, we can show that also the isomorphism problem for automatic linear orders is Σ^1_1-complete. This answers Question 4.3 (and consequently Question 4.7) from [22] negatively. From this proof, we obtain two further results as well:

- Elementary equivalence of automatic linear orders is Π^0_1-complete (giving another partial answer to Question 4.8 from [22]).
- The isomorphism problem for linear orders of the form $(L; \leq_{\mathsf{lex}})$, where L is a deterministic context-free language and $\leq_{\mathsf{lex}}$ is the lexicographic ordering on strings, is Σ^1_1-complete.

The latter result answers a question of Ésik [10], where he proved undecidability of the isomorphism problem for linear orders of the form $(L; \leq_{\mathsf{lex}})$ with L a context-free language and $\leq_{\mathsf{lex}}$ the lexicographic ordering. The same problem is decidable for regular languages instead of deterministic context-free languages [36] (an exponential time algorithm for the case that the regular language is given by a deterministic automaton can be obtained from [2]).

For the important subclass of automatic scattered linear orders, we can provide a reduction of the isomorphism problem to true arithmetic. Although non-arithmetical, true arithmetic belongs to a relatively low level of the hyperarithmetical hierarchy. This shows that the isomorphism problem for scattered linear orders is strictly simpler than for general linear orders. It remains open, whether the isomorphism problem for scattered linear orders is decidable. It seems that our new techniques cannot help to solve this problem: Our proof for linear orders uses shuffle sums, and this construction always yields non-scattered linear orders.

Related work. Beyond the works cited so far, we should mention that Blumensath and Grädel [4] were the first to prove undecidability of the isomorphism problem for automatic structures. The paper [19] studies several methods of constructing automatic equivalence structures with different types of isomorphism invariants. Automatic linear orders were also studied in unpublished work by Delhommé [7], where it was shown that an ordinal is automatic if and only if it is strictly below ω^ω. Extending Delhommé's technique, Khoussainov et al. [24] prove that all automatic linear orders have finite Hausdorff ranks. In the same paper [24], automatic order trees are studied and it is shown that the Cantor-Bendixson rank of any automatic order tree is finite. The paper [20] constructs automatic structures of high ordinal height and Scott rank, again by using transition graphs of Turing-machines. Recently, we extended our techniques to ω-automatic structures (which are represented by Büchi-automata instead of ordinary finite automata) [25], where we proved that the isomorphism problem for ω-automatic structures (even ω-automatic trees of finite height) does not belong to the analytical hierarchy. Lastly, we mention that for equivalence structures, linear orders, and order trees which have automatic presentations over a unary alphabet, the isomorphism problem is decidable in polynomial time [27].

2 Preliminaries

Let $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$. Let $p(x_1, \ldots, x_n) \in \mathbb{N}[x_1, \ldots, x_n]$ be a polynomial with non-negative integer coefficients. We define

$$\mathsf{Img}_+(p) = \{p(y_1, \ldots, y_n) \mid y_1, \ldots, y_n \in \mathbb{N}_+\}.$$

If p is not the zero-polynomial, then $\mathsf{Img}_+(p) \subseteq \mathbb{N}_+$.

2.1 Logic

A *relational structure* $\mathcal{S}$ consists of a *domain* D and finitely many atomic relations on the set D. It will be denoted by $(D; R_1, \ldots, R_n)$, where $R_1, \ldots, R_n$ are the atomic relations of $\mathcal{S}$. The *signature* of $\mathcal{S}$ is the tuple consisting of the arities of all relations R_i. We will only consider structures with

countable domains. Occasionally, we will write $a \in \mathcal{S}$ for $a \in D$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two structures with the same signature and with disjoint domains, then we write $\mathcal{S}_1 \uplus \mathcal{S}_2$ for the union of the two structures. Hence, when writing $\mathcal{S}_1 \uplus \mathcal{S}_2$, we implicitly express that the domains of $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint. More generally, if $\{\mathcal{S}_i \mid i \in I\}$ is a class of pairwise disjoint structures with the same signature, then we denote with $\biguplus\{\mathcal{S}_i \mid i \in I\}$ the union of these structures.

We assume that the reader has some familiarity with first-order logic (briefly FO) and second-order logic, see e.g. [15] for more details. Recall that in second-order logic there are first-order variables (denoted by lower case letters) and second-order variables of arbitrary arity (denoted by upper case letters), which can be quantified existentially and universally. First-order variables range over elements of the domain of the underlying structure, whereas second-order variables range over relations of the appropriate arity. With $\mathcal{S} \models \varphi$ we denote the fact that the formula φ evaluates to true in the structure $\mathcal{S}$ (which has to have the appropriate signature); if φ has free variables then appropriate values have to get assigned to these variables. If $\varphi(x)$ is an FO-formula, x is a first-order variable, and $m \in \mathbb{N}$, then we will also allow the formulas $\exists^{\geq m} x : \varphi(x)$ and $\exists^{=m} x : \varphi(x)$ with the following meanings: $\mathcal{S} \models \exists^{\geq m} x : \varphi(x)$ (resp., $\mathcal{S} \models \exists^{=m} x : \varphi(x)$) if there exist at least (resp., exactly) m many elements $a \in \mathcal{S}$ with $\mathcal{S} \models \varphi(a)$. Clearly these quantifiers do not increase the expressiveness of FO. Moreover, the quantifier $\exists^{\geq m} x$ can be replaced by a block of m ordinary existential quantifiers.

Definition 1. *Two structures $\mathcal{S}$ and $\mathcal{S}'$ are* elementary equivalent *(denoted $\mathcal{S} \equiv \mathcal{S}'$) if for all first-order formulas φ without free variables, we have:*

$$\mathcal{S} \models \varphi \iff \mathcal{S}' \models \varphi\,.$$

FSO (for "fragment of second order logic") [26] is a proper extension of FO by the following three formation rules ($\mathcal{S}$ is again a structure with the appropriate signature):

- If $\varphi(x)$ is an FSO-formula and x a first-order variable, then $\exists^{=\infty} x : \varphi$ is also an FSO-formula, which has the following meaning: $\mathcal{S} \models \exists^{=\infty} x : \varphi(x)$ if and only if there are infinitely many $a \in \mathcal{S}$ with $\mathcal{S} \models \varphi(a)$.
- If $\varphi(x)$ is an FSO-formula, x a first-order variable, and $p \in \mathbb{N}_+$, then $\exists^{(p)} x\, \varphi$ is also an FSO-formula, which has the following meaning $\mathcal{S} \models \exists^{(p)} x\, \varphi$ if and only if the number of $a \in \mathcal{S}$ with $\mathcal{S} \models \varphi(a)$ is finite and a multiple of p.
- If X is an n-ary second-order variable that occurs only negatively (i.e., in the range of an odd number of negations) in the FSO-formula $\varphi(X)$, then $\exists X$ infinite : $\varphi(X)$ is also an FSO-formula, which has the following meaning: $\mathcal{S} \models \exists X$ infinite : $\varphi(X)$ if and only if there exists an infinite relation $R \subseteq \mathcal{S}^n$ such that $\mathcal{S} \models \varphi(R)$.[3]

Example 2. FSO allows to express that a graph with edge relation E has an infinite clique by the formula

$$\exists X \text{ infinite} \forall x, y : x, y \in X \wedge x \neq y \rightarrow (x, y) \in E$$

(X is a unary second-order variable). Note that this formula is equivalent to

$$\exists X \text{ infinite} \forall x, y : x \notin X \vee y \notin X \vee x = y \vee (x, y) \in E,$$

i.e., X occurs indeed only negatively.

2.2 Computability

We assume some familiarity with the basic concepts of computability theory, see e.g. [30,35] for more details. An *index* for a computable function f is a Turing machine (or the Gödel number of a Turing machine) that computes f. With Σ^0_n we denote the n^{th} (existential) level of the

[3] The condition that X occurs only negatively in $\varphi(X)$ ensures that if $\mathcal{S} \models \varphi(R)$ then $\mathcal{S} \models \varphi(Q)$ for every subset $Q \subseteq R$.

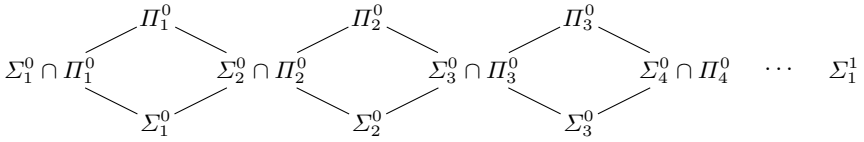


Fig. 1. The arithmetical hierarchy and Σ_1^1

arithmetical hierarchy; it is the class of all subsets $A \subseteq \mathbb{N}$ such that there exists a computable predicate $P \subseteq \mathbb{N}^{n+1}$ with

$$A = \{a \in \mathbb{N} \mid \exists x_1 \forall x_2 \cdots Q x_n : (a, x_1, \ldots, x_n) \in P\}, \tag{1}$$

where $Q = \exists$ ($Q = \forall$) for n odd (even). The set of complements of Σ_n^0-sets is denoted by Π_n^0. The *arithmetical hierarchy* is $\bigcup_{n\geq 0} \Sigma_n^0 = \bigcup_{n\geq 0} \Pi_n^0$. There is a lot of freedom in the definition of the classes Σ_n^0 and Π_n^0. Instead of requiring P in (1) to be a recursive predicate, one can take a quantifier-free FO-formula over the structure $(\mathbb{N}; +, \times)$. Moreover, each of the quantifiers in (1) can be replaced by a block of quantifiers of the same type (since a tuple of naturals can be encoded by a single natural using a computable coding function). In particular, the quantifier $\exists^{\geq m} x$ is allowed in place of an ordinary existential quantifier.[4] Similarly, one may allow first-order quantifiers that range over finite objects of a particular type, e.g., words from a regular language, finite automata, computable mappings (represented by Turing machines), etc.. On the other hand, if quantifiers over arbitrary subsets of natural numbers are allowed, one moves from the arithmetical hierarchy to the so called *analytical hierarchy*. We will only need the first (existential) level Σ_1^1 of this hierarchy. It is the class of all subsets of $\mathbb{N}$ of the form $\{n \in \mathbb{N} \mid \exists A \subseteq \mathbb{N} : (\mathbb{N}; +, \times) \models \varphi(A, n)\}$, where $\varphi(A, n)$ is an FO-formula (the subset A is used as an additional unary predicate). Figure 1 shows an inclusion diagram. By fixing some effective encoding of strings by natural numbers, we can talk about Σ_n^0-sets, Π_n^0-sets, and Σ_1^1-sets of strings over an arbitrary alphabet. Statements about the completeness of a set for one of the above classes will always refer to many-one reductions. A typical example of a Σ_1^1-set, which does not belong to the arithmetical hierarchy and which is not Σ_1^1-complete is *true arithmetic*, i.e., the first-order theory of $(\mathbb{N}; +, \times)$, which we denote by $\mathsf{FOTh}(\mathbb{N}; +, \times)$. In terms of the hyperarithmetical hierarchy,[5] $\mathsf{FOTh}(\mathbb{N}; +, \times)$ is complete for its first non-arithmetical level Δ_ω^0.

2.3 Automata

We assume basic terminologies and notations from automata theory, see, for example, [18]. As usual, we denote with Σ^+ the set of all non-empty words on the alphabet Σ. For a fixed alphabet Σ, a *(non-deterministic finite) automaton* is a tuple $\mathcal{A} = (S, \Delta, I, F)$ where S is the finite set of states, $\Delta \subseteq S \times \Sigma \times S$ is the transition relation, $I \subseteq S$ is a set of initial states, and $F \subseteq S$ is the set of accepting states. A *run* of $\mathcal{A}$ on a word $u = a_1 a_2 \cdots a_n$ ($a_1, a_2 \ldots, a_n \in \Sigma$) is a word over Δ of the form $r = (q_0, a_1, q_1)(q_1, a_2, q_2) \cdots (q_{n-1}, a_n, q_n)$. It is *accepting* if $q_0 \in I$ and $q_n \in F$. For technical reasons, we only consider runs on non-empty words. For $u \in \Sigma^+$, we denote by $\mathsf{Run}(\mathcal{A}, u) \subseteq \Delta^+$ the set of all accepting runs of $\mathcal{A}$ on the word u. We define $L_+(\mathcal{A}) = \{u \in \Sigma^+ \mid \mathsf{Run}(\mathcal{A}, u) \neq \emptyset\}$, it is the set of all non-empty words accepted by $\mathcal{A}$. Thus the language $L(\mathcal{A})$ accepted by $\mathcal{A}$ is $L_+(\mathcal{A})$ if $I \cap F = \emptyset$, otherwise it is $L_+(\mathcal{A}) \cup \{\varepsilon\}$. Let $\mathsf{Run}(\mathcal{A}) = \bigcup_{u \in \Sigma^+} \mathsf{Run}(\mathcal{A}, u)$ be the set of all accepting runs of $\mathcal{A}$. This is a regular language: A finite automaton for $\mathsf{Run}(\mathcal{A})$ can be obtained by replacing every transition $(p, a, q) \in \Delta$ by $(p, (p, a, q), q)$. In addition, if $I \cap F \neq \emptyset$, we have to intersect with Δ^+. For the accepting run $r \in \mathsf{Run}(\mathcal{A})$, we write $\mathrm{lab}(r)$ for the word accepted by

[4] On the other hand, the exact counting quantifier $\exists^{=m} x$ introduces an additional quantifier alternation and therefore does not preserve the levels of the arithmetical hierarchy.

[5] The hyperarithmetical hierarchy is a kind of transfinite extension of the arithmetical hierarchy. The class of all hyperarithmetical sets is $\Sigma_1^1 \cap \Pi_1^1$, where Π_1^1 is the set of complements of Σ_1^1-sets. See [30] for more details.

r, i.e., $r \in \mathsf{Run}(\mathcal{A}, \mathrm{lab}(r))$. We state the following fact which is used at several occasions in the paper: for a given automaton $\mathcal{A}$, one can compute effectively the cardinality $|L(\mathcal{A})| \in \mathbb{N} \cup \{\aleph_0\}$ of the language accepted by $\mathcal{A}$.

We use *synchronous n-tape automata* to recognize n-ary relations. Such automata have n input tapes, each of which contains one of the input words. The n tapes are read in parallel until all input words are processed. Formally, let $\Sigma_\diamond = \Sigma \cup \{\diamond\}$ where $\diamond \notin \Sigma$. For words $w_1, w_2, \ldots, w_n \in \Sigma^*$, their *convolution* $w_1 \otimes \cdots \otimes w_n$ or $\otimes(w_1, \ldots, w_n)$ is a word in $(\Sigma_\diamond^n)^*$ of length $\max\{|w_1|, \ldots, |w_n|\}$, and the k^{th} symbol of $w_1 \otimes \cdots \otimes w_n$ is $(\sigma_1, \ldots, \sigma_n)$ where σ_i is the k^{th} symbol of w_i if $k \leq |w_i|$, and $\sigma_i = \diamond$ otherwise. We lift this definition to sets of words in the obvious way, i.e., for languages $L_1, L_2 \subseteq \Sigma^*$ let $L_1 \otimes L_2 = \{w_1 \otimes w_2 \mid w_1 \in L_1, w_2 \in L_2\}$. For an n-ary relation $R \subseteq (\Sigma^*)^n$, let $R^\otimes$ be the set of convolutions of tuples from R. Then R is *FA recognizable* if $R^\otimes$ is a regular language.

2.4 Automatic structures

A structure $\mathcal{S}$ is called *automatic* over Σ if its domain is a regular subset of Σ^* and each of its atomic relations is FA recognizable; any tuple $\mathcal{P}$ of automata that accept the domain and the relations of $\mathcal{S}$ is called an *automatic presentation of* $\mathcal{S}$; in this case, we write $\mathcal{S}(\mathcal{P})$ for $\mathcal{S}$. If an automatic structure $\mathcal{S}$ is isomorphic to a structure $\mathcal{S}'$, then $\mathcal{S}$ is called an *automatic copy* of $\mathcal{S}'$ and $\mathcal{S}'$ is *automatically presentable*. In this paper we sometimes abuse the terminology referring to $\mathcal{S}'$ as simply automatic and calling an automatic presentation of $\mathcal{S}$ also automatic presentation of $\mathcal{S}'$. We also simplify our statements by saying "given/compute an automatic structure $\mathcal{S}$" for "given/compute an automatic presentation $\mathcal{P}$ of a structure $\mathcal{S}(\mathcal{P})$".

Example 3. The following structures are known to be automatic:

- Finite structures
- $(\mathbb{N}; \leq, +)$
- $(\mathbb{Q}; \leq)$
- Every ordinal $< \omega^\omega$
- Transition graphs of Turing machines, i.e., graphs where the set of nodes is the set of all configurations of a fixed Turing machine M and there is an edge from configuration c_1 to configuration c_2 if M can move in one step from c_1 to c_2.

On the other hand, the following structures have no automatic copies:

- $(\mathbb{N}; \times)$ [3]
- Every ordinal $\geq \omega^\omega$ [7]
- Every infinite field [23]
- $(\mathbb{Q}; +)$ [37]

Well-known examples of automatic linear orders are the *lexicographic order* $\leq_{\mathsf{lex}}$ and the *length-lexicographic order* $\leq_{\mathsf{llex}}$ on a regular language D. To define them, we first need a fixed linear order $\leq$ on the alphabet Σ of D. For $w, w' \in D$, we say that w is *lexicographically less* than w', denoted by $w <_{\mathsf{lex}} w'$, if either w is a proper prefix of w' or there exist $x, y, z \in \Sigma^*$ and $\sigma, \tau \in \Sigma$ such that $w = x\sigma y$, $w' = x\tau z$, and $\sigma < \tau$. We write $w \leq_{\mathsf{lex}} w'$ if either $w = w'$ or $w <_{\mathsf{lex}} w'$. Furthermore, $w \leq_{\mathsf{llex}} w'$ if $|w| < |w'|$ or ($|w| = |w'|$ and $w \leq_{\mathsf{lex}} w'$). For convenience, in this paper, we use $\leq_{\mathsf{lex}}$ to denote the lexicographic order regardless of the corresponding alphabets and orders on the alphabets. The precise definition of $\leq_{\mathsf{lex}}$ in different occurrences will be clear from the context. Note that $\leq_{\mathsf{llex}}$ is always a well-order. Hence, every automatic structure can be expanded by a well-order on its domain.

The following theorem from [3,16,21,26,32] lays out the main motivation for investigating automatic structures.

Theorem 4. *From an automatic presentation $\mathcal{P}$ and an FSO-formula $\varphi(x_1, \ldots, x_n)$ in the signature of $\mathcal{S}(\mathcal{P})$, one can compute an automaton for the language $\{(v_1, \ldots, v_n) \in \mathcal{S}(\mathcal{P})^n \mid \mathcal{S}(\mathcal{P}) \models \varphi(v_1, \ldots, v_n)\}^\otimes$. In particular:*

– *The* FSO *theory of any automatic structure* $\mathcal{S}$ *is (uniformly) decidable.*
– *If* $\mathcal{S}$ *is automatic and* $\mathcal{S}'$ *is* FSO*-interpretable in* $\mathcal{S}'$*, then* $\mathcal{S}'$ *is effectively automatic (i.e., an automatic presentation for* $\mathcal{S}'$ *can be computed from an automatic presentation for* $\mathcal{S}$*).*

This paper is mainly interested in the complexity of the following two decision problems:

Definition 5. *For a class* $\mathcal{K}$ *of automatic presentations, we consider the following sets:*

– *The* isomorphism problem $\mathsf{Iso}(\mathcal{K})$ *for* $\mathcal{K}$ *is the set of pairs* $(\mathcal{P}_1, \mathcal{P}_2) \in \mathcal{K} \times \mathcal{K}$ *of automatic presentations with* $\mathcal{S}(\mathcal{P}_1) \cong \mathcal{S}(\mathcal{P}_2)$.
– *The* elementary equivalence problem $\mathsf{EE}(\mathcal{K})$ *for* $\mathcal{K}$ *is the set of pairs* $(\mathcal{P}_1, \mathcal{P}_2) \in \mathcal{K} \times \mathcal{K}$ *of automatic presentations such that* $\mathcal{S}(\mathcal{P}_1) \equiv \mathcal{S}(\mathcal{P}_2)$.

If $\mathcal{K}$ is the class of all automatic presentations for a class $\mathcal{C}$ of relational structures (e.g. trees or linear orders), then we will briefly speak of the isomorphism problem for (automatic members of) $\mathcal{C}$. The classes $\mathcal{C}$ that will appear in this paper (equivalence relations and various classes of trees and linear orders) have the nice property that they can be axiomatized by a single FSO-formula. Theorem 4 implies that the corresponding classes $\mathcal{K}$ of automatic presentations are decidable. In this case, since the set of all FO-formulas without free variables can be enumerated, Theorem 4 implies that $\mathsf{EE}(\mathcal{K}) \in \Pi^0_1$. On the other hand, the isomorphism problem can be much harder: The isomorphism problem for the class of all automatic structures is complete for Σ^1_1 [23]. However, if one restricts to special subclasses of automatic structures, this complexity bound can be reduced. For example, for the class of automatic ordinals [24] and also the class of automatic Boolean algebras [23], the isomorphism problem is decidable. Another interesting result is that the isomorphism problem for locally finite automatic graphs is Π^0_3-complete [32].

3 Automatic Equivalence Structures

An equivalence structure is of the form $\mathcal{E} = (D; \approx)$ where $\approx$ is an equivalence relation on D. As usual, we denote with $[a]_\approx$ (or briefly $[a]$, if $\approx$ is clear from the context), the equivalence class containing $a \in D$. In this section, we prove that the isomorphism problem for automatic equivalence structures is Π^0_1-complete. This result also follows from our handling of automatic trees in Section 4. However, we present it separately here as it is a good starting point for introducing our techniques.

Let $\mathcal{E} = (D; \approx)$ be an automatic equivalence structure. Define the function $h_\mathcal{E} : \mathbb{N}_+ \cup \{\aleph_0\} \to \mathbb{N} \cup \{\aleph_0\}$ such that for all $n_+ \in \mathbb{N} \cup \{\aleph_0\}$, $h_\mathcal{E}(n)$ equals the number of equivalence classes (possibly infinite) in $\mathcal{E}$ of size n. Note that for given $n \in \mathbb{N}_+ \cup \{\aleph_0\}$, the value $h_\mathcal{E}(n)$ can be computed effectively: it equals m if

$$\mathcal{E} \models \exists^{=m} x : (\exists^{=n} y : x \approx y \wedge \forall y : x \approx y \to x \leq_{\mathsf{llex}} y)$$

(here $\exists^{=\aleph_0}$ stands for $\exists^{=\infty}$). Thus, one can check by Theorem 4 whether $h_\mathcal{E}(n) = m$ for $m = \aleph_0, 0, 1, 2, \ldots$ until one finds the correct value. Given two automatic equivalence structures $\mathcal{E}_1$ and $\mathcal{E}_2$, deciding if $\mathcal{E}_1 \cong \mathcal{E}_2$ amounts to checking if $h_{\mathcal{E}_1} = h_{\mathcal{E}_2}$, i.e., if $\forall n \in \mathbb{N}_+ \cup \{\aleph_0\} : h_{\mathcal{E}_1}(n) = h_{\mathcal{E}_2}(n)$. Since the function $h_\mathcal{E}$ is computable for automatic equivalence structures, the isomorphism problem for automatic equivalence structures is consequently in Π^0_1.

Consider the automatic equivalence structures $(a^+; =)$ and $((bb)^+; =)$ that are isomorphic. There is no FA recognizable isomorphism, but there is a computable one (mapping, e.g., a^n to b^{2n}). This is no coincidence as the following proposition shows.

Proposition 6. *Let* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *be two isomorphic automatic equivalence structures. Then there exists a computable isomorphism from* $\mathcal{E}_1$ *to* $\mathcal{E}_2$*.*

Proof. Let $\mathcal{E}_i = (V_i; \approx_i)$ (w.l.o.g. $V_1 \cap V_2 = \emptyset$) and let $\leq_{\mathsf{llex}}$ denote the length-lexicographic order on $V_1 \cup V_2$. In the following, $\min(U)$ ($U \subseteq V_i$) denotes the minimal element of U w.r.t. $\leq_{\mathsf{llex}}$. This

minimum exists, since $\leq_{\mathsf{llex}}$ is a well-order. Let $\mathsf{Min}_i = \{u \in V_i \mid u = \min([u])\}$. This set is FSO-definable in the structure $(V_i; \approx_i, \leq_{\mathsf{llex}})$. Since $\leq_{\mathsf{llex}}$ is FA recognizable, this structure is automatic. Hence Min_i is a regular language. It contains a unique element from each equivalence class of $\approx_i$. For $u \in V_i$, define:

$$\begin{aligned} m_1(u) &= |\{v \in V_i \mid v <_{\mathsf{llex}} u \wedge u \approx_i v\}| \in \mathbb{N} \\ m_2(u) &= |\{x \in \mathsf{Min}_i \mid x <_{\mathsf{llex}} \min([u]) \wedge |[x]| = |[u]|\}| \in \mathbb{N} \\ m_3(u) &= |[u]| \in \mathbb{N}_+ \cup \{\aleph_0\} \end{aligned}$$

Thus, $m_1(u)$ is the number of equivalent, but smaller words than u and $m_2(u)$ is the number of equivalence classes of size $|[u]|$ whose minimal element is smaller than the minimal element of $[u]$. Clearly, every isomorphism f between $\mathcal{E}_1$ and $\mathcal{E}_2$ must satisfy $m_3(u) = m_3(f(u))$ for all $u \in V_1$. Moreover, there exists a unique isomorphism f such that $m_1(u) = m_1(f(u))$ and $m_2(u) = m_2(f(u))$ for all $u \in V_1$. Below, we show that the mappings m_i $(1 \leq i \leq 3)$ are computable. This implies that the unique isomorphism f with $m_i(u) = m_i(f(u))$ for all $1 \leq i \leq 3$, $u \in V_1$ is computable as well: For a given $u \in V_1$ we enumerate all $v \in V_2$ until we find a v with $m_i(u) = m_i(v)$ for $1 \leq i \leq 3$. Then, we set $f(u) = v$.

Computability of $m_3(u)$ follows from the fact that $[u]$ is FSO-definable in the automatic structure $(V_i; \approx_i, \leq_{\mathsf{llex}}, \{u\})$. Hence $[u]$ is effectively regular (i.e., an automaton for $[u]$ can be computed from u), and we can compute the cardinality of $[u]$. Let $\kappa \in \mathbb{N}_+ \cup \{\aleph_0\}$ be this cardinality. The cardinality of the set $\{v \in V_i \mid v <_{\mathsf{llex}} u \wedge u \approx_i v\}$ (i.e., $m_1(u)$) can be computed by the same argument. Finally, using the cardinality $\kappa = |[u]|$, we can find an FSO-definition for the set $\{x \in \mathsf{Min}_i \mid x <_{\mathsf{llex}} \min([u]) \wedge |[x]| = \kappa\}$ in the structure $(V_i; \approx_i, \leq_{\mathsf{llex}}, \mathsf{Min}_i)$, which is again automatic. Hence the set $\{x \in \mathsf{Min}_i \mid x <_{\mathsf{llex}} \min([u]) \wedge |[x]| = \kappa\}$ is regular as well and we can compute its cardinality (which is $m_2(u)$). □

For the Π_1^0 lower bound, we use a reduction from Hilbert's 10^{th} problem: Given a polynomial $p(x_1, \ldots, x_k) \in \mathbb{Z}[x_1, \ldots, x_k]$, decide whether the equation $p(x_1, \ldots, x_k) = 0$ has a solution in $\mathbb{N}_+$ (for technical reasons, it is useful to exclude 0 in solutions). This problem is well-known to be undecidable, see e.g. [28]. More precisely, let $X \subseteq \mathbb{N}_+$ be some Σ_1^0-complete set. Then, Matiyasevich provides two polynomials $p_1(x, x_1, \ldots, x_k), p_2(x, x_1, \ldots, x_k) \in \mathbb{N}[x, x_1, \ldots, x_k]$ such that for all $n \in \mathbb{N}_+$: $n \in X$ if and only if $\exists y_1, \ldots, y_k \in \mathbb{N}_+ : p_1(n, y_1, \ldots, y_k) - p_2(n, y_1, \ldots, y_k) = 0$, i.e., $p_1(n, y_1, \ldots, y_k) = p_2(n, y_1, \ldots, y_k)$. Hence the mapping $n \mapsto (p_1(n, x_1, \ldots, x_k), p_2(n, x_1, \ldots, x_k))$ is a reduction of X to the set

$$\{(q_1, q_2) \in \mathbb{N}[x_1, \ldots, x_k]^2 \mid k \in \mathbb{N}_+, \exists \overline{c} \in \mathbb{N}_+^k : q_1(\overline{c}) = q_2(\overline{c})\}.$$

Since this set belongs to Σ_1^0, it is therefore Σ_1^0-complete. Hence, the set

$$\{(p_1, p_2) \in \mathbb{N}[x_1, \ldots, x_k]^2 \mid k \in \mathbb{N}_+, \forall \overline{c} \in \mathbb{N}_+^k : p_1(\overline{c}) \neq p_2(\overline{c})\}$$

is Π_1^0-complete.

The following constructions prepare Lemma 7 which connects polynomials from $\mathbb{N}[\overline{x}]$ with automata. For a symbol a, let Σ_k^a denote the alphabet

$$\Sigma_k^a = \{a, \diamond\}^k \setminus \{(\diamond, \ldots, \diamond)\}$$

and let σ_i denote the i^{th} component of $\sigma \in \Sigma_k^a$. For $\overline{e} = (e_1, \ldots, e_k) \in \mathbb{N}_+^k$, let us define the word

$$a^{\overline{e}} = a^{e_1} \otimes a^{e_2} \otimes \cdots \otimes a^{e_k} \in (\Sigma_k^a)^*. \tag{2}$$

For a language L, let

$$\otimes_k (L) = \{u_1 \otimes u_2 \otimes \cdots \otimes u_k \mid u_1, \ldots, u_k \in L\}. \tag{3}$$

Lemma 7. *There exists an algorithm that, given a non-zero polynomial $p(\overline{x}) \in \mathbb{N}[\overline{x}]$ in k variables, constructs an automaton $\mathcal{A}[p(\overline{x})]$ on the alphabet Σ_k^a with $L_+(\mathcal{A}[p(\overline{x})]) = \otimes_k(a^+)$ such that, for all $\overline{c} \in \mathbb{N}_+^k$, $p(\overline{c})$ equals the number $|\mathsf{Run}(\mathcal{A}, a^{\overline{c}})|$ of accepting runs of $\mathcal{A}$ on $a^{\overline{c}}$.*

Proof. We first build by induction on the construction of the polynomial p an automaton $\mathcal{A}_p$ with $p(\overline{c}) = |\mathsf{Run}(\mathcal{A}_p, a^{\overline{c}})$ for all $\overline{c} \in \mathbb{N}_+^k$: The base case is provided by the polynomials 1 and x_i.

Let $\mathcal{A}_1$ be a deterministic automaton accepting $\otimes_k(a^+)$. Next, suppose $p(x_1, \ldots, x_k) = x_i$ for some $i \in \{1, \ldots, k\}$. Define the automaton $\mathcal{A}_{x_i} = (\{q_1, q_2\}, \Delta, \{q_1\}, \{q_2\})$, where

$$\Delta = \{(q_1, \sigma, q_j) \mid j \in \{1, 2\}, \sigma \in \Sigma_k^a, \sigma_i = a\} \cup \{(q_2, \sigma, q_2) \mid \sigma \in \Sigma_k^a\}.$$

When the automaton $\mathcal{A}_{x_i}$ runs on an input word $a^{\overline{c}}$, it has exactly c_i many times the chance to move from state q_1 to the final state q_2. Therefore there are exactly $c_i = p(\overline{c})$ many accepting runs on $a^{\overline{c}}$.

Next, let $p_1(\overline{x})$ and $p_2(\overline{x})$ be polynomials in $\mathbb{N}[\overline{x}]$. Assume as inductive hypothesis that there is, for $i \in \{1, 2\}$, an automaton $\mathcal{A}_{p_i} = (S_i, \Delta_i, I_i, F_i)$ such that the number of accepting runs of $\mathcal{A}_{p_i}$ on $a^{\overline{c}}$ equals $p_i(\overline{c})$.

For the polynomial $p(\overline{x}) = p_1(\overline{x}) + p_2(\overline{x})$, let $\mathcal{A}_p$ be the disjoint union of the automata $\mathcal{A}_{p_1}$ and $\mathcal{A}_{p_2}$. Then, the number of accepting runs of $\mathcal{A}_p$ on $a^{\overline{c}}$ is $p_1(\overline{c}) + p_2(\overline{c})$.

For the polynomial $p(\overline{x}) = p_1(\overline{x}) \cdot p_2(\overline{x})$, consider the Cartesian product $\mathcal{A}_p = \mathcal{A}_{p_1} \times \mathcal{A}_{p_2}$ of the automata $\mathcal{A}_{p_1}$ and $\mathcal{A}_{p_2}$. This Cartesian product is the automaton $(S_1 \times S_2, \Delta, I_1 \times I_2, F_1 \times F_2)$, where

$$\Delta = \{((p_1, p_2), \sigma, (q_1, q_2)) \mid (p_1, \sigma, q_1) \in \Delta_1, (p_2, \sigma, q_2) \in \Delta_2\}.$$

Then, the number of accepting runs of $\mathcal{A}_p$ on $a^{\overline{c}}$ is $p_1(\overline{c}) \cdot p_2(\overline{c})$.

Finally, to ensure that $L_+(\mathcal{A}[p(\overline{x})]) = \otimes_k(a^+)$, let $\mathcal{A}[p(\overline{x})]$ be the Cartesian product $\mathcal{A}_p \times \mathcal{A}_1$. □

We next construct, from a non-zero polynomial $p(\overline{x}) \in \mathbb{N}[x_1, \ldots, x_k]$ an automatic equivalence structure $\mathcal{E}(p)$. Let the automaton $\mathcal{A} = \mathcal{A}[p(\overline{x})]$ satisfy the properties guaranteed by Lemma 7 and recall that $\mathsf{Run}(\mathcal{A})$ is the set of accepting runs of $\mathcal{A}$. The domain of the automatic equivalence structure $\mathcal{E}(p)$ is $\mathsf{Run}(\mathcal{A})$. Moreover, two words $u, v \in \mathsf{Run}(\mathcal{A})$ are equivalent if and only if $\mathrm{lab}(u) = \mathrm{lab}(v)$. By definition and Lemma 7, a natural number $y \in \mathbb{N}_+$ belongs to $\mathsf{Img}_+(p)$ if and only if there exists a word $u \in L_+(\mathcal{A})$ with precisely y accepting runs, if and only if $\mathcal{E}(p)$ contains an equivalence class of size y, i.e., $\mathsf{Img}_+(p) = \{n \in \mathbb{N}_+ \cup \{\aleph_0\} : h_{\mathcal{E}(p)}(n) > 0\}$.

It is well known that the polynomial function $C : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with

$$C(x, y) = (x + y)^2 + 3x + y \tag{4}$$

is injective ($C(x, y)/2$ is the position of $(x + y, x)$ in the lexicographic enumeration of $\mathbb{N}^2$). In the following, let $\mathcal{E}_{\mathsf{Good}}$ denote the countably infinite equivalence structure with

$$h_{\mathcal{E}_{\mathsf{Good}}}(n) = \begin{cases} \infty & \text{if } n \in \{C(y, z) \mid y, z \in \mathbb{N}_+, y \neq z\} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 8. *The set of automatic presentations $\mathcal{P}$ with $\mathcal{S}(\mathcal{P}) \cong \mathcal{E}_{\mathsf{Good}}$ is hard for Π_1^0. It equals the set of automatic presentations $\mathcal{P}$ with $\mathcal{S}(\mathcal{P}) \equiv \mathcal{E}_{\mathsf{Good}}$.*

Proof. For non-zero polynomials $p_1(\overline{x}), p_2(\overline{x}) \in \mathbb{N}[x_1, \ldots, x_k]$, define the following three (non-zero) polynomials from $\mathbb{N}[x_1, \ldots, x_k]$ (with $k \geq 2$):

$$S_1(\overline{x}) = C(p_1(\overline{x}), p_2(\overline{x})), \qquad S_2(\overline{x}) = C(x_1 + x_2, x_1), \qquad S_3(\overline{x}) = C(x_1, x_1 + x_2).$$

Let $\mathcal{E}(S_1)$, $\mathcal{E}(S_2)$, and $\mathcal{E}(S_3)$ be the automatic equivalence structures corresponding to these polynomials according to the above definition. Finally, let $\mathcal{E}$ be the disjoint union of $\aleph_0$ many copies of these three equivalence structures. Then $\mathcal{E}$ is effectively automatic.

If $p_1(\overline{c}) = p_2(\overline{c})$ for some $\overline{c} \in \mathbb{N}_+^k$, then there is $y \in \mathbb{N}_+$ such that $C(y, y) \in \mathsf{Img}_+(S_1)$. Therefore in $\mathcal{E}$ there is an equivalence class of size $C(y, y)$ and no such equivalence class exists in $\mathcal{E}_{\mathsf{Good}}$. Hence $\mathcal{E} \not\cong \mathcal{E}_{\mathsf{Good}}$.

Conversely, suppose that $p_1(\overline{c}) \neq p_2(\overline{c})$ for all $\overline{c} \in \mathbb{N}_+^k$. For all $y, z \in \mathbb{N}_+$, $\mathcal{E}$ contains an equivalence class of size $C(y, z)$ if and only if $C(y, z)$ belongs to $\mathsf{Img}_+(S_1) \cup \mathsf{Img}_+(S_2) \cup \mathsf{Img}_+(S_3)$,

if and only if $y \neq z$, if and only if $\mathcal{E}_{\mathsf{Good}}$ contains an equivalence class of size $C(y,z)$. Therefore, for any $s \in \mathbb{N}_+$, $\mathcal{E}$ contains an equivalence class of size s if and only if $\mathcal{E}_{\mathsf{Good}}$ contains an equivalence class of size s. Since moreover, $\mathcal{E}_{\mathsf{Good}}$ has an equivalence class of size s if and only if it has $\aleph_0$ many equivalence classes of size s, and similarly for $\mathcal{E}$, we get $\mathcal{E} \cong \mathcal{E}_{\mathsf{Good}}$.

In summary, we have reduced the Π^0_1-hard problem

$$\{(p_1(\overline{x}), p_2(\overline{x})) \in \mathbb{N}[x_1, \ldots, x_k]^2 \mid k \geq 2, \forall \overline{c} \in \mathbb{N}_+^k : p_1(\overline{c}) \neq p_2(\overline{c})\}$$

to the set of automatic presentations of $\mathcal{E}_{\mathsf{Good}}$. Hence the first statement is proved.

The second statement follows since all equivalence classes of $\mathcal{E}_{\mathsf{Good}}$ and $\mathcal{E}$ are finite. This implies that $\mathcal{E}_{\mathsf{Good}} \equiv \mathcal{E}$ if and only if $\mathcal{E}_{\mathsf{Good}} \cong \mathcal{E}$. □

Theorem 9. *The isomorphism problem and the elementary equivalence problem for automatic equivalence structures are Π^0_1-complete.*

Proof. At the beginning of this section, we already argued that the isomorphism problem is in Π^0_1; hardness follows immediately from Proposition 8, since $\mathcal{E}_{\mathsf{Good}}$ is necessarily automatic.

Note that the set of automatic presentations of equivalence structures is decidable. Hence the elementary equivalence problem belongs to Π^0_1 as explained after Definition 1. Hardness follows from the second statement of Proposition 8. □

4 Automatic Trees

A *forest* is a structure $T = (V; \leq)$, where $\leq$ is a partial order on V such that for every $x \in V$, the order $\leq$ restricted to the set $\{y \mid y \leq x\}$ of ancestors of x is a finite linear order. A *tree* is a forest with least element, called *root*, or the empty structure. Let us fix a forest $H = (V; \leq)$. For a node $v \in V$ we denote with $H(u)$ (the *subtree of H rooted at u*) the forest H restricted to the set $\{v \in V \mid u \leq v\}$; this is indeed a tree. The *level* of a node $v \in V$ is $|\{x \mid x < v\}| \in \mathbb{N}$. The *height* of H is the supremum of the levels of all nodes in V; it may be infinite even in case H is well-founded. One may also view H as a directed graph $(V; E)$, where there is an edge $(u, v) \in E$ if and only if u is the largest element in $\{x \mid x < v\}$. We call E the *edge relation* of H. The edge relation E is FO-definable in $(V; \leq)$: $(u, v) \in E$ if and only if $u < v$ and $\neg\exists x : u < x < v$. The set of *children* of $v \in V$ is denoted by $E(v) = \{u \in V \mid (v, u) \in E\}$. The forest $H = (V; \leq)$ is *well-founded*, if there does not exist an *infinite branch* in H, which is an infinite sequence of nodes $v_0 < v_1 < v_2 < \cdots$. An example of a well-founded tree is $(X^{<k}; \preceq)$ where $X^{<k}$ denotes the set of all words over the set X of length at most $k-1$ and $\preceq$ is the prefix order on these words. We will denote this tree briefly by $X^{<k}$. We also write $X^{<\omega}$ for the tree whose universe is X^* with the prefix relation. A forest $H_1 = (V_1; \leq_1)$ *embeds* into a forest $H_2 = (V_1; \leq_1)$, briefly $H_1 \hookrightarrow H_2$, if there exists an injective mapping $f : V_1 \to V_2$ such that for all $u, v \in V_1$, $u \leq_1 v$ if and only if $f(u) \leq_2 f(v)$.

Let us emphasize again that in this paper, forests and in particular trees are partial orders. In the literature, a tree viewed as a partial order is also called an *order trees*, whereas the graph consisting of the edge relation of an order tree is also called a *successor tree*. Hence, when talking about trees, we implicitly speak of order trees.

Note that for an automatic forest H and a node $v \in H$ the subtree $H(v)$ is FO-definable in H (using v as a parameter). Hence, the domain of $H(v)$ is a regular subset of the domain of H and $H(v)$ is effectively automatic (i.e., an automatic presentation for $H(v)$ can be computed from an automatic presentation for H and v) as well.

We use $\mathcal{T}_n$ ($n \in \mathbb{N}$) to denote the class of all automatic presentations $\mathcal{P}$ such that $\mathcal{S}(\mathcal{P})$ is an automatic tree of height at most n. Note that one can write down a sentence of FSO that is satisfied by a directed graph G if and only if G is a forest. Hence the set of all automatic presentations of forests is decidable by Theorem 4. The same argument shows decidability of the

set of all automatic presentations of trees, of well-founded trees[6], and of trees of height at most n. The same holds for the class of trees of finite height:

Theorem 10. *The set of automatic presentations of trees of finite height is decidable.*

Proof. Let $\mathcal{P}$ be an automatic presentation of some tree and let $\mathcal{S}(\mathcal{P}) = (V; \leq)$. Then the relation $\geq$ is rational[7] and a rational transducer $\mathcal{A}$ for $\geq$ can be computed from $\mathcal{P}$. The tree $\mathcal{S}(\mathcal{P})$ has finite height if and only if this transducer is finite-valued, i.e., there exists $n \in \mathbb{N}$ such that $|\{v \in V \mid u \geq v\}| \leq n$ for all words $u \in V$. But this is decidable by [38]. □

4.1 Upper bounds for trees with countably many infinite branches

We will show that the isomorphism problem for automatic trees with countably many infinite branches can be reduced to true arithmetic. Towards this aim, we will parameterize these trees T by their *embeddability rank* $\mathsf{erank}(T)$ (which is defined in Section 4.1.1) and show in Section 4.1.2 the arithmetical upper bound Π^0_{2k-4} for trees of embeddability rank at most k. The claim then follows from the uniformity of our proof and the computability of $\mathsf{erank}(T)$.

4.1.1 The embeddability rank of a tree Let T be a tree. Its *embeddability rank* or *e-rank* $\mathsf{erank}(T)$ is defined to be

$$\mathsf{erank}(T) = \sup\{k+1 \mid k \in \mathbb{N} \cup \{\omega\}, \mathbb{N}^{<k} \hookrightarrow T\} \in \mathbb{N}_+ \cup \{\omega, \omega+1\}\,.$$

Then the empty tree has e-rank 1 (since only the empty tree $\mathbb{N}^{<0}$ can be embedded into it), any finite and non-empty tree has e-rank 2, the disjoint union of all trees $\mathbb{N}^{<k}$ for $k \in \mathbb{N}$ together with a new root has e-rank ω, and $\mathbb{N}^{<\omega}$ has e-rank $\omega+1$. A nonempty tree has e-rank 2 if and only if it does not contain an infinite antichain, i.e., if and only if it is finitely branching and has only finitely many branching points, i.e., nodes with at least two children.

By $\mathcal{T}^{\mathrm{er}}_k$, we denote the set of all automatic presentations of trees of e-rank at most $k \in \mathbb{N}_+ \cup \{\omega, \omega+1\}$. Then we have obviously

$$\mathcal{T}^{\mathrm{er}}_1 \subsetneq \mathcal{T}^{\mathrm{er}}_2 \subsetneq \mathcal{T}^{\mathrm{er}}_3 \subsetneq \cdots \subsetneq \bigcup_{i \geq 1} \mathcal{T}^{\mathrm{er}}_i \subseteq \mathcal{T}^{\mathrm{er}}_\omega \subsetneq \mathcal{T}^{\mathrm{er}}_{\omega+1} \tag{5}$$

(strictness in the last inclusion holds since the automatic tree $\mathbb{N}^{<\omega}$ has e-rank $\omega+1$). The aim of this section is to prove that an automatic tree has only countably many infinite branches if and only if its e-rank is finite. For this, we first prove that the e-rank of a tree is $\omega+1$ if and only if it has uncountably many infinite branches (Lemma 11) and then, that no automatic tree has e-rank ω (Lemma 14). This latter result implies in particular $\bigcup_{i \geq 1} \mathcal{T}^{\mathrm{er}}_i = \mathcal{T}^{\mathrm{er}}_\omega$. Thus, the inclusion chain (5) can be simplified to

$$\mathcal{T}^{\mathrm{er}}_1 \subsetneq \mathcal{T}^{\mathrm{er}}_2 \subsetneq \mathcal{T}^{\mathrm{er}}_3 \subsetneq \cdots \subsetneq \bigcup_{i \geq 1} \mathcal{T}^{\mathrm{er}}_i = \mathcal{T}^{\mathrm{er}}_\omega \subsetneq \mathcal{T}^{\mathrm{er}}_{\omega+1}$$

We will also show that all the classes $\mathcal{T}^{\mathrm{er}}_k$ ($k \in \mathbb{N}_+ \cup \{\omega, \omega+1\}$) are decidable.

Lemma 11. *Let $T = (L; \leq)$ be a countable tree. The following are equivalent:*

(1) $\mathsf{erank}(T) = \omega + 1$
(2) T has $2^{\aleph_0}$ many infinite branches.

[6] This latter result was first shown in [24]. Note that a tree $(V; \leq)$ is well-founded if and only if the undirected graph $(V; \leq \cup \geq)$ does not contain an infinite clique. Hence, one can use the FSO-formula from Example 2.

[7] A relation $R \subseteq \Gamma^* \times \Gamma^*$ is rational, if it can be accepted by a rational transducer, which is a finite automaton with transition labels from $\Gamma^* \times \Gamma^*$.

(3) T has uncountably many infinite branches.

Proof. First suppose that $\mathsf{erank}(T) = \omega + 1$. Then the tree $\mathbb{N}^{<\omega}$ embeds into T, hence T has $2^{\aleph_0}$ many infinite branches which proves the implication (1)⇒(2). The implication (2)⇒(3) is trivial.

To prove the remaining implication (3)⇒(1), suppose that T has uncountably many infinite branches. For a node $x \in L$, let $\mathrm{br}(x)$ denote the number of infinite branches of $T(x)$. Let $B = \{x \in L \mid \mathrm{br}(x) > \aleph_0\}$. We first show that B contains two incomparable nodes (w.r.t. the tree order $\leq$). Suppose towards a contradiction that B is linearly ordered. Let X denote the set of nodes $y \in L \setminus B$ whose immediate predecessor (i.e., father) belongs to B. Since L is countable, so is X. Hence the number of infinite branches of T equals

$$1 + \sum_{y \in X} \mathrm{br}(y) \leq 1 + \aleph_0 \cdot \aleph_0 = \aleph_0$$

contradicting our assumption that T contains uncountably many many infinite branches. By induction, it follows that the complete binary tree $\{0,1\}^{<\omega}$ can be embedded into T. But then we have $\mathbb{N}^{<\omega} \hookrightarrow \{0,1\}^{<\omega} \hookrightarrow T$, which is statement (1). □

By the following result, the properties from Lemma 11 are decidable.

Proposition 12. *The set of automatic presentations of trees with only countably many infinite branches is decidable.*

Proof. Let $T = (L; \leq)$ be an automatic tree. Let $B \subseteq 2^L$ be the set of its infinite branches, and let in be the set of pairs $(x, a) \in L \times B$ with $x \in a$. In [26], it was shown that the structure $(L \cup B; \leq, B, \mathsf{in})$ is effectively ω-automatic. Hence, by [1], its $(\mathsf{FO} + \exists^{2^{\aleph_0}})$-theory[8] is decidable. In this logic, it is expressible that the set B has size $2^{\aleph_0}$, which means that T has $2^{\aleph_0}$ many infinite branches. By Lemma 11, this is equivalent to the fact that T has uncountably many infinite branches. □

Our next aim is to show that there is no automatic tree of e-rank ω. As a first step, we show that no well-founded automatic tree has e-rank ω.

Lemma 13. *Let $T = (L; \leq)$ be an automatic well-founded tree. Then $\mathsf{erank}(T)$ is finite.*

Proof. Let us fix an arbitrary length-lexicographical ordering $\leq_{\mathsf{llex}}$ on the set of words L. We define the *Kleene-Brouwer ordering* $\mathsf{KB}(T) = (L; \sqsubseteq)$, where $u \sqsubset v$ if and only if $u \leq v$ or there exist $w, u_1, v_1 \in L$ such that $u_1 \leq u$, $v_1 \leq v$, w is the parent node of u_1 and of v_1 in T, and $u_1 <_{\mathsf{llex}} v_1$. This order is linear. Moreover, since T is well-founded, $\mathsf{KB}(T)$ is an ordinal.

Note that the expansion of T by $\leq_{\mathsf{llex}}$ is still an automatic structure and that $\sqsubseteq$ is first-order definable in this structure. Hence $\mathsf{KB}(T)$ is an automatic ordinal. Thus, by [8], there exists $k \in \mathbb{N}$ with $\mathsf{KB}(T) < \omega^k$.

By induction on i, we now show $\mathsf{KB}(T(x)) \geq \omega^i$ in case $\mathbb{N}^{<i+1} \hookrightarrow T(x)$ for all nodes x. If $\mathbb{N}^{<2} \hookrightarrow T$ then T is infinite and we get $\mathsf{KB}(T) \geq \omega$. Now assume that $i \geq 2$ and that $\mathbb{N}^{<i+1} \hookrightarrow T$. Then there exists an infinite antichain $\{a_0, a_1, a_2, \ldots\}$ in T such that $\mathbb{N}^{<i} \hookrightarrow T(a_j)$ for all $j \geq 0$. W.l.o.g. assume that $a_0 \sqsubset a_1 \sqsubset a_2 \sqsubset \cdots$. By induction, we have $\mathsf{KB}(T(a_j)) \geq \omega^{i-1}$ for all $j \geq 0$ and therefore

$$\mathsf{KB}(T) \geq \sum_{j \in \mathbb{N}} \mathsf{KB}(T(a_j)) \geq \omega^{i-1} \cdot \omega = \omega^i .$$

This finishes the induction.

Since $\mathsf{KB}(T) < \omega^k$, we therefore get $\mathbb{N}^{<k+1} \not\hookrightarrow T$, i.e., the e-rank of T is at most $k+1$ and therefore finite. □

[8] $\mathsf{FO} + \exists^{2^{\aleph_0}}$ is the extension of FO by the quantifier $\exists^{2^{\aleph_0}}$, which expresses that there are $2^{\aleph_0}$ many elements with a given property.

By Lemma 13, no well-founded automatic tree has e-rank ω. To prove this fact for all automatic trees, we will use the notion of Cantor-Bendixon-rank of a tree $T = (L; \leq)$: Let $d(T)$ denote the restriction of T to those nodes that belong to at least two infinite branches of T. This is again a countable tree (possibly empty). By [24], there exists a number $r \in \mathbb{N}$ with $d^r(T) = d^{r+1}(T)$. The least such number is called the *Cantor-Bendixon-rank*. Note that it is very different from the e-rank we defined above: the tree $\mathbb{N}^{<\omega}$ has Cantor-Bendixon-rank 0 and e-rank $\omega + 1$. The following lemma generalizes Lemma 13.

Lemma 14. *Let $T = (L; \leq)$ be an automatic tree with countably many infinite branches. Then $\mathsf{erank}(T)$ is finite.*

Proof. The lemma is shown by induction on the Cantor-Bendixon-rank of T. If this rank equals 0, then every node of T belongs to at least two infinite branches, so T is either empty or embeds $\{0,1\}^{<\omega}$. Since T has only countably many infinite branches, we get $T = \emptyset$. Hence, $\mathsf{erank}(T) = 1$.

Now suppose that the Cantor-Bendixon-rank of $T = (L; \leq)$ is $r + 1$. We split L into three sets: L_0 contains all nodes that do not belong to any infinite branch, L_1 consists of those nodes that belong to precisely one infinite branch, and L_2 is the rest (i.e., $(L_2; \leq) \cong d(T)$). The sets L_1 and L_2 (and therefore L_0) are effectively regular [23]. Let T_0 be obtained from the forest $(L_0; \leq)$ by adding a new root. Then T_0 is a well-founded automatic tree that has finite e-rank e_0 by Lemma 13. Also $d(T)$ is an automatic tree with at most $\aleph_0$ many infinite branches. Since its Cantor-Bendixon-rank is properly smaller than that of T, the induction hypothesis guarantees that its e-rank e_2 is finite.

We want to show that the e-rank of T is at most $e_2 + e_0 + 1$. So let $k \in \mathbb{N}_+$ and let $f : \mathbb{N}^{<k} \hookrightarrow T$ be an embedding. We have to prove $k \leq e_2 + e_0$. If the image of f is contained in L_2, then f is an embedding into $d(T)$ implying $k < e_2 \leq e_2 + e_0$. Otherwise let $w_2 \in \mathbb{N}^{<k}$ be a word of minimal length with $f(w_2) \notin L_2$. Then all words of length $< |w_2|$ are mapped into L_2, i.e., the restriction of f to $\mathbb{N}^{<|w_2|}$ is an embedding into $d(L)$ which implies $|w_2| < e_2$. We now distinguish two cases.

(a) Suppose $f(w_2) \in L_0$. Then the mapping $g : \mathbb{N}^{<k-|w_2|} \to T_0$ with $g(x) = f(w_2 x)$ is an embedding implying $k - |w_2| < e_0$ and therefore $k < e_2 + e_0$.
(b) Now suppose $f(w_2) \in L_1$. If $|w_2| = k - 1$, then $|w_2| < e_2$ implies $k \leq e_2 \leq e_2 + e_0$. So let $|w_2| < k - 1$. Since $(L_1; \leq)$ is a disjoint union of copies of ω, there is some $n \in \mathbb{N}$ with $f(w_2 n) \in L_0$. As in (a), we obtain $k - |w_2 n| < e_0$ which, together with $|w_2 n| = |w_2| + 1 \leq e_2$ implies $k < e_2 + e_0$. ☐

Corollary 15. *An automatic tree T has countably many infinite branches if and only if $\mathsf{erank}(T)$ is finite.*

Proof. If T has countably many infinite branches, then $\mathsf{erank}(T)$ is finite by Lemma 14. If $\mathsf{erank}(T)$ is finite, then it is not $\omega + 1$ and so T has only countably many infinite branches by Lemma 11. ☐

We finish our consideration of the e-rank proving that it can be computed. Consider the following recursively defined formulas of FSO for $k \in \mathbb{N}_+$:

$$\begin{aligned} \mathsf{rank}_1(x) &= (x = x) \\ \mathsf{rank}_{k+1}(x) &= \exists X \text{ infinite } \forall y, y' \in X : x < y \wedge (y \leq y' \to y = y') \wedge \mathsf{rank}_k(y)\,. \end{aligned}$$

By induction on k, one can show:

Lemma 16. *Let T be some tree, v a node of T, and $k \in \mathbb{N}$. Then $T \models \mathsf{rank}_{k+1}(v)$ if and only if $\mathbb{N}^{<k} \hookrightarrow T(v)$ (i.e., $\mathsf{erank}(T(v)) > k$).*

Corollary 17. *For a given automatic tree, one can compute its e-rank.*

Proof. Let $T = (L; \leq)$ be an automatic tree. First, we check whether $\mathsf{erank}(T) = \omega + 1$: By Lemma 11, we have to check whether T contains $2^{\aleph_0}$ many infinite branches, which is decidable by Proposition 12.

Next, assume that it turns out that $\mathsf{erank}(T) < \omega + 1$. Thus, by Lemma 15, $\mathsf{erank}(T)$ is finite. Then, for every $k \in \mathbb{N}$, $\mathsf{erank}(T) \leq k$ if and only if $T \models \neg\exists x : \mathsf{rank}_{k+1}(x)$. Since this is a sentence of FSO that can be computed from k, we can check effectively, whether $\mathsf{erank}(T) \leq k$. By doing this successively for $k = 1, 2, \ldots$, we can compute $\mathsf{erank}(T)$. ☐

4.1.2 The isomorphism problem Note that the empty tree is the only tree of e-rank 1. The following definition will be used in our proof of an upper bound for the isomorphism problem for automatic trees of higher e-rank.

Definition 18. *Let $T = (L; \leq)$ be some tree of e-rank $k \geq 2$. Then the* initial segment $I(T) \subseteq L$ *consists of all nodes $x \in L$ with $\mathsf{erank}(T(x)) = k$.*

Note that the root of T always belongs to the initial segment $I(T)$ and that $x \leq y \in I(T)$ implies $x \in I(T)$.

Lemma 19. *Let $T = (V; \leq)$ be some tree with $\mathsf{erank}(T) = k \geq 2$. Then $(I(T); \leq)$ is a tree of e-rank 2.*

Proof. Since $I(T)$ is downwards closed in T, $(I(T); \leq)$ is indeed a tree itself. Since $I(T) \neq \emptyset$ by the above remark, we get $\mathsf{erank}(I(T); \leq) \geq 2$. If $\mathsf{erank}(I(T); \leq) \geq 3$, then $\mathbb{N}^{<2}$ would embed into $(I(T); \leq)$, i.e., $I(T)$ would contain an infinite antichain $B = \{b_i \mid i \in \mathbb{N}\}$. Since the e-rank of $T(b_i)$ is k, we get $\mathbb{N}^{<k-1} \hookrightarrow T(b_i)$ implying $\mathbb{N}^{<k} \hookrightarrow T$. But this contradicts $\mathsf{erank}(T) = k$. □

We now study the isomorphism problem $\mathsf{Iso}(\mathcal{T}_2^{\mathrm{er}})$ of automatic trees of e-rank at most 2. Recall that a nonempty tree has e-rank 2 if and only if it is finitely branching and has only finitely many branching points. But this is the case if and only if it is a finite tree where some of the leaves are replaced by infinite branches. In particular, there are only finitely many isomorphisms between two trees of e-rank 2. But these isomorphisms need not be automatic in case the two trees are automatic (e.g., consider the automatic trees $\{a\}^{<\omega}$ and $\{aa\}^{<\omega}$ that are both isomorphic to ω).

Lemma 20. *The following holds:*

(1) Any isomorphism between two automatic trees of e-rank 2 is computable.
(2) From two automatic presentations of trees of e-rank 2, one can compute a list of all (indices of) isomorphisms.
(3) The isomorphism problem $\mathsf{Iso}(\mathcal{T}_2^{\mathrm{er}})$ of automatic trees of e-rank at most 2 is decidable.

Proof. Clearly, (2) implies (3). For (1) and (2), let $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{T}_2^{\mathrm{er}}$ and let $T_i = \mathcal{S}(\mathcal{P}_i) = (L_i; \leq_i)$. Let $B_i \subseteq L_i$ be the set of nodes $x \in L_i$ such that there exists a leaf y or a branching point y in T_i with $x \leq_i y$. Let $C_i \subseteq L_i$ be the union of B_i and all children of nodes in B_i. Clearly, C_i is downwards closed in T_i, i.e., $(C_i, \leq_i)$ is a tree. Since T_i has no infinite antichains (otherwise, it would embed $\mathbb{N}^{<2}$ and therefore have e-rank at least 3), the sets C_1 and C_2 are finite and computable from $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively.

Any isomorphism $f : T_1 \to T_2$ induces an isomorphism $g : (C_1; \leq_1) \to (C_2; \leq_2)$. Note that the nodes from $C_i \setminus B_i$ are the starting points of non-branching infinite branches of T_i. Hence, conversely, any isomorphism $g : (C_1; \leq_1) \to (C_2; \leq_2)$ extends uniquely to an isomorphism $f : T_1 \to T_2$. Given the finite set g, the isomorphism f is even computable: for $x \in C_1$, output $g(x)$; for $x \in L_1 \setminus C_1$, compute the unique node $y \in C_1 \setminus B_1$ with $y <_1 x$, compute the distance in T_1 from y to x, and map x to the unique node of the same distance in T_2 from $g(y)$.

A list of all indices of isomorphisms from T_1 to T_2 can be computed by listing all isomorphisms between the finite trees $(C_1; \leq_1)$ and $(C_2; \leq_2)$. By the above argument we can compute from an isomorphism $g : (C_1; \leq_1) \to (C_2; \leq_2)$ an index for the unique isomorphism $f : T_1 \to T_2$ that extends g. This shows (2). □

Lemma 21. *From an automatic presentation $\mathcal{P} \in \mathcal{T}_2^{\mathrm{er}}$ of a tree of e-rank at most 2, one can compute a first-order sentence $\varphi_{\mathcal{P}}$ such that, for all trees T, we have*

$$T \models \varphi_{\mathcal{P}} \iff T \cong \mathcal{S}(\mathcal{P}) .$$

Proof. Recall that a tree has e-rank at most 2 if and only if it is a finite tree where some of the leaves are replaced by infinite branches. Hence any tree of e-rank at most 2 can be described in first-order logic up to isomorphism. To find $\varphi_{\mathcal{P}}$, simply list all sentences that describe trees of e-rank at most 2 and output the first from this list that holds in $\mathcal{S}(\mathcal{P})$. □

Lemma 22. *For $3 \leq n < \omega$, the isomorphism problem $\mathsf{Iso}(\mathcal{T}_n^{\mathrm{er}})$ for automatic trees of e-rank at most n belongs to Π^0_{2n-5}.*

Proof. The proof proceeds by induction on n. So let $n \geq 3$, and $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{T}_n^{\mathrm{er}}$, $T_i = \mathcal{S}(\mathcal{P}_i) = (L_i; \leq_i)$. Define the automatic forest $H = T_1 \uplus T_2$ and let E be the edge relation of H (it is again automatic). Moreover, let $I = I(T_1) \uplus I(T_2)$.

For $x \in H$ and a tree t, let $\#(x,t) \in \mathbb{N} \cup \{\aleph_0\}$ denote the number of children $y \in E(x)$ of x in H such that $H(y) \cong t$. Given this definition, we have $T_1 \cong T_2$ if and only if there exists an isomorphism $f : (I(T_1), \leq_1) \to (I(T_2), \leq_2)$ such that for all $x \in I(T_1)$:

$$\forall \text{ trees } t \text{ with } \mathsf{erank}(t) \leq n-1 : \#(x,t) = \#(f(x),t)\,.$$

If t is not automatic, then $\#(x,t) = \#(f(x),t) = 0$, i.e., we can restrict quantification to automatic trees of e-rank at most $n-1$. Hence, $T_1 \cong T_2$ if and only if one of the isomorphisms $f : (I(T_1); \leq_1) \to (I(T_2); \leq_2)$ satisfies the following:

$$\forall x \in I(T_1)\ \forall \mathcal{P} \in \mathcal{T}_{n-1}^{\mathrm{er}}\ \forall \ell \geq 1 \left(\begin{array}{ll} & \exists^{\geq \ell} x \in E(x) \setminus I : \quad \mathcal{S}(\mathcal{P}) \cong H(x) \\ \iff & \exists^{\geq \ell} x \in E(f(x)) \setminus I : \mathcal{S}(\mathcal{P}) \cong H(x) \end{array} \right) \tag{6}$$

Recall that by Lemma 19 and 20, a finite list of all isomorphisms $f : (I(T_1), \leq_1) \to (I(T_2), \leq_2)$ (each of these isomorphisms is computable) can be computed. Hence, it suffices to show that the formula (6) is a Π^0_{2n-5}-statement. Note that we have $\mathsf{erank}(\mathcal{S}(\mathcal{P})) < n$ and $\mathsf{erank}(H(x)) < n$ for all $\mathcal{P} \in \mathcal{T}_{n-1}^{\mathrm{er}}$ and for all nodes $x \in (E(x) \setminus I) \cup (E(f(x)) \setminus I)$. We now distinguish the cases $n = 3$ and $n > 3$:

- Let $n = 3$. Then the subformula $\mathcal{S}(\mathcal{P}) \cong H(x)$ in (6) is equivalent to $H(x) \models \varphi_{\mathcal{P}}$, where $\varphi_{\mathcal{P}}$ is the FSO-formula from Lemma 21. By Lemma 16, the set I is FSO-definable and therefore effectively regular. Given $x \in I(T_1)$, the set $E(x)$ is also effectively regular. Since the isomorphism f is computable, the set $E(f(x))$ is effectively regular as well. Hence the equivalence in brackets is an FSO-statement about an automatic structure and therefore decidable. Thus, the whole formula belongs to $\Pi^0_1 = \Pi^0_{2n-5}$.
- Now let $n > 3$. Hence the subformula $\mathcal{S}(\mathcal{P}) \cong H(x)$ is by the induction hypothesis a Π^0_{2n-7} statement. As for the case $n = 3$, one can argue that the sets I, $E(x)$, and $E(f(x))$ are effectively regular. Thus, the whole formula belongs to Π^0_{2n-5}. □

Proposition 23. *The isomorphism problem $\mathsf{Iso}(\mathcal{T}_\omega^{\mathrm{er}})$ of automatic trees with countably many infinite branches is many-one reducible to $\mathsf{FOTh}(\mathbb{N}; +, \times)$.*

Proof. Let $P_1, P_2 \in \mathcal{T}_\omega^{\mathrm{er}}$ be automatic presentations of trees T_1 and T_2 with countably many branches. Then there are $k_1, k_2 \in \mathbb{N}$ such that $\mathsf{erank}(T_i) = k_i$. By Corollary 17, these natural numbers can be computed. If $k_1 \neq k_2$, then the two trees are not isomorphic. Otherwise, they are isomorphic if and only if (P_1, P_2) belongs to the Π_{2k_1-4}-relation Iso_{k_1}. The uniformity of the proof of Lemma 22 implies the result. □

Note that a tree of height n has e-rank at most $n+2$. Hence, we have $\mathcal{T}_n \subseteq \mathcal{T}_{n+2}^{\mathrm{er}}$ for all $n \geq 0$ (recall that $\mathcal{T}_n$ is the class of all automatic presentations of trees of height at most n). Lemma 22 implies that the isomorphism problem $\mathsf{Iso}(\mathcal{T}_n)$ for automatic trees of height at most n belongs to $\Pi^0_{2(n+2)-5} = \Pi^0_{2n-1}$ for all $n \geq 1$. We can improve this upper bound by two levels:

Lemma 24. *The isomorphism problem for the class $\mathcal{T}_n$ of automatic trees of height at most n is*

- *decidable for $n = 1$ and*
- *in Π^0_{2n-3} for all $n \geq 2$.*

Proof. For the first part of the lemma, take $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{T}_1$. To check if $\mathcal{S}(\mathcal{P}_1) \cong \mathcal{S}(\mathcal{P}_2)$, it suffices to compute the cardinality of the two trees, which can be done as their universes are regular languages.

We show the second part of the lemma (i.e., where $n \geq 2$) by induction on n. Consider automatic presentations $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{T}_n$. Define the automatic forest $H = \mathcal{S}(\mathcal{P}_1) \uplus \mathcal{S}(\mathcal{P}_2)$ and let E be the edge relation of H (it is again automatic). Let r_i be the root of T_i.

For $n = 2$, the trees T_1 and T_2 have height at most 2. Then $T_1 \cong T_2$ if and only if

$$\forall \kappa \in \mathbb{N} \cup \{\aleph_0\}\ \forall \ell \geq 1 \left(\begin{array}{c} \exists^{\geq \ell} x \in E(r_1) : |E(x)| = \kappa \\ \iff \exists^{\geq \ell} x \in E(r_2) : |E(x)| = \kappa \end{array} \right).$$

In other words: for every $\kappa \in \mathbb{N} \cup \{\aleph_0\}$, r_1 and r_2 have the same number of children with exactly κ children. For fixed $\kappa \in \mathbb{N} \cup \{\aleph_0\}$ and $\ell \geq 1$, the equivalence in brackets is an FSO-sentence and therefore decidable by Theorem 4. Thus, the whole formula is indeed a Π^0_1-sentence (note that $2n - 3 = 1$ for $n = 2$).

Now assume that the statement holds for $n - 1$. We have $T_1 \cong T_2$ if and only if

$$\forall v \in E(r_1) \cup E(r_2)\ \forall \ell \geq 1 \left(\begin{array}{c} \exists^{\geq \ell} x \in E(r_1) : H(v) \cong H(x) \\ \iff \exists^{\geq \ell} x \in E(r_2) : H(v) \cong H(x) \end{array} \right).$$

By quantifying over all $v \in E(r_1) \cup E(r_2)$, we quantify over all isomorphism types of trees that occur as a subtree rooted at a child of r_1 or r_2. For each of these isomorphism types τ, we express that r_1 and r_2 have the same number of children x with $H(x)$ of type τ. Note that the automatic trees $H(v)$ and $H(x)$ in the above formula have height $n-1$. Hence, there exists a Π^0_{2n-5}-statement for $H(v) \cong H(x)$. Thus, $T_1 \cong T_2$ is a Π^0_{2n-3}-statement. □

4.2 Arithmetical lower bounds for trees of finite height

In this section, we will show that the upper bounds from Lemmas 22 and 24 are optimal. For the minimal classes $\mathcal{T}_3^{\mathrm{er}}$ and $\mathcal{T}_2$, this is an immediate consequence of Proposition 8:

Corollary 25. *There exists an automatic tree T_{Good} of height 2 and e-rank 3 such that the set of automatic presentations $\mathcal{P}$ with $\mathcal{S}(\mathcal{P}) \cong T_{\mathsf{Good}}$ is Π^0_1-hard. Hence, the isomorphism and the elementary equivalence problems for the classes $\mathcal{T}_2$ and $\mathcal{T}_3^{\mathrm{er}}$ are Π^0_1-hard.*

Proof. Let $\mathcal{E} = (L; \equiv)$ be an automatic equivalence structure without infinite equivalence classes. Now build the tree $T(\mathcal{E})$ as follows:

- The set of nodes is $L \cup \{r\} \cup \{ua \mid u \in L, u \text{ is } \leq_{\mathsf{llex}}\text{-minimal in } [u]_\equiv\}$ where r and a are two new letters.
- r is the root, its children are the words ending in a, and the children of ua are the words from $[u]_\equiv$.

Then it is clear that $T(\mathcal{E})$ is an automatic tree of height at most 2. Since any node of the form ua has only finitely many successors, it has e-rank 3 (since $\mathbb{N}^{<3}$ does not embed). Furthermore, an automatic presentation for $T(\mathcal{E})$ can be computed from one for $\mathcal{E}$.

Recall the automatic equivalence structure $\mathcal{E}_{\mathsf{Good}}$ (which does not have infinite equivalence classes) from Section 3. Note that $\mathcal{E} \cong \mathcal{E}_{\mathsf{Good}}$ if and only if $T(\mathcal{E}) \cong T(\mathcal{E}_{\mathsf{Good}})$. Hence, we reduced the set of automatic presentations of $\mathcal{E}_{\mathsf{Good}}$ to the set of automatic presentations of $T(\mathcal{E}_{\mathsf{Good}})$. Since the former is Π^0_1-hard by Proposition 8, so is the latter.

For $m, n \in \mathbb{N}$, consider the following FO-formula:

$$\varphi_{m,n} \ = \ \exists r \left(\forall x : (x, r) \notin E \ \wedge \ \exists^{\geq m} y \left((r, y) \in E \ \wedge \ \exists^{=n} z : (y, z) \in E \right) \right).$$

Then, if $\mathcal{E}$ is an equivalence structure without infinite equivalence classes, we have $T(\mathcal{E}) \cong T(\mathcal{E}_{\mathsf{Good}})$ if and only if

$$\forall m, n \in \mathbb{N} : T(\mathcal{E}) \models \varphi_{m,n} \iff T(\mathcal{E}_{\mathsf{Good}}) \models \varphi_{m,n}$$

(here, the relation symbol E from the formula $\varphi_{m,n}$ denotes the edge relation of $T(\mathcal{E})$ and $T(\mathcal{E}_{\mathsf{Good}})$, respectively). Hence, $T(\mathcal{E}) \cong T(\mathcal{E}_{\mathsf{Good}})$ if and only if $T(\mathcal{E}) \equiv T(\mathcal{E}_{\mathsf{Good}})$. Thus, the set of automatic presentations of trees elementary equivalent to $T(\mathcal{E}_{\mathsf{Good}})$ is Π^0_1-hard as well. □

In the rest of this section we will prove a generalization of Corollary 25: The isomorphism problem for the class of automatic trees of height at most $n \geq 2$ is Π^0_{2n-3}-hard. Note that with Lemma 24 it follows that this problem is Π^0_{2n-3}-complete. To prove Π^0_{2n-3}-hardness, we provide a generic reduction from an arbitrary Π^0_{2n-3}-predicate $P_n(x_0)$ to the isomorphism problem for $\mathcal{T}_n$.

In the following lemma and its proof, all quantifiers with unspecified range run over $\mathbb{N}_+$. The lemma states the existence of a certain normal form for Π^0_{2n-3}-predicates, which will be needed later in this section.

Lemma 26. *From a Π^0_{2n-3}-predicate $P_n(x_0)$, one can compute Π^0_{2i-3}-predicates*

$$P_i(x_0, x_1, y_1, x_2, y_2, \ldots, x_{n-i}, y_{n-i})$$

for $2 \leq i < n$ such that

(i) $P_{i+1}(\overline{x})$ is logically equivalent to $\forall x_{n-i} \exists y_{n-i} : P_i(\overline{x}, x_{n-i}, y_{n-i})$ for $2 \leq i < n$ and
(ii) $\forall y_{n-i} : \neg P_i(\overline{x}, x_{n-i}, y_{n-i})$ implies $\forall x'_{n-i} \geq x_{n-i}\ \forall y_{n-i} : \neg P_i(\overline{x}, x'_{n-i}, y_{n-i})$,

where $\overline{x} = (x_0, x_1, y_1, \ldots, x_{n-i-1}, y_{n-i-1})$.

Proof. The predicates P_i are constructed by induction, starting with $i = n-1$ down to $i = 2$ where the construction of P_i does not assume that (i) or (ii) hold true for P_{i+1}.

So let $2 \leq i < n$ such that $P_{i+1}(\overline{x})$ is a $\Pi^0_{2(i+1)-3}$-predicate. Then there exists a Π^0_{2i-3}-predicate $P(\overline{x}, x_{n-i}, y_{n-i})$ such that $P_{i+1}(\overline{x})$ is logically equivalent to

$$\forall x_{n-i} \exists y_{n-i} : P(\overline{x}, x_{n-i}, y_{n-i})\,.$$

But this is logically equivalent to

$$\forall x_{n-i}\ \forall x'_{n-i} \leq x_{n-i}\ \exists y_{n-i} : P(\overline{x}, x'_{n-i}, y_{n-i})\,. \tag{7}$$

Let $\varphi(\overline{x}, x_{n-i})$ be

$$\forall x'_{n-i} \leq x_{n-i}\ \exists y_{n-i} : P(\overline{x}, x'_{n-i}, y_{n-i})\,.$$

Then for any $x_{n-i} \in \mathbb{N}_+$,

$$\neg\varphi(\overline{x}, x_{n-i}) \implies \forall x \geq x_{n-i} : \neg\varphi(\overline{x}, x)\,. \tag{8}$$

Since $\forall x'_{n-i} \leq x_{n-i}$ is a bounded quantifier, the formula $\varphi(\overline{x}, x_{n-i})$ belongs to Σ^0_{2i-2} (see for example [35, p. 61]). Thus, there is a Π^0_{2i-3}-predicate $P_i(\overline{x}, x_{n-i}, y_{n-i})$ such that

$$\varphi(\overline{x}, x_{n-i}) \iff \exists y_{n-i} : P_i(\overline{x}, x_{n-i}, y_{n-i})\,. \tag{9}$$

Therefore (7) (and consequently $P_{i+1}(\overline{x})$) is logically equivalent to $\forall x_{n-i}\ \exists y_{n-i} : P_i(\overline{x}, x_{n-i}, y_{n-i})$. Moreover,

$$\begin{aligned}
\forall y_{n-i} : \neg P_i(\overline{x}, x_{n-i}, y_{n-i}) &\overset{(9)}{\iff} \neg\varphi(\overline{x}, x_{n-i}) \\
&\overset{(8)}{\implies} \forall x \geq x_{n-i} : \neg\varphi(\overline{x}, x) \\
&\overset{(9)}{\iff} \forall x \geq x_{n-i}\ \forall y_{n-i} : \neg P_i(\overline{x}, x, y_{n-i})\,.
\end{aligned}$$

This shows (ii) from the lemma. □

Let us fix the predicates P_i from Lemma 26 for the rest of Section 4. By induction on $2 \leq i \leq n$, we will construct the following trees of height i and e-rank $i+1$:

- test trees $T^i_{\overline{c}} \in \mathcal{T}_i$ for $\overline{c} \in \mathbb{N}_+^{1+2(n-i)}$ (which depend on P_i) and
- trees $U^i_\kappa \in \mathcal{T}_i$ for $\kappa \in \mathbb{N}_+ \cup \{\omega\}$.

The crucial properties of these trees are the following, where $\overline{c} \in \mathbb{N}_+^{1+2(n-i)}$:

(P1) $P_i(\overline{c})$ holds if and only if $T^i_{\overline{c}} \cong U^i_\omega$.
(P2) $P_i(\overline{c})$ does not hold if and only if $T^i_{\overline{c}} \cong U^i_m$ for some $m \in \mathbb{N}_+$.

For $3 \leq i \leq n$, the idea is that $T^i_{\overline{c}} \cong U^i_\kappa$ if and only if

$$\kappa = \inf(\{\omega\} \cup \{x_{n-i+1} \mid \forall y_{n-i+1} \in \mathbb{N}_+ : \neg P_{i-1}(\overline{c}, x_{n-i+1}, y_{n-i+1})\}) .$$

Property (P1) is certainly sufficient for proving Π^0_{2n-3}-hardness (with $i = n$), the second property (P2) and therefore the trees U^i_m for $m < \omega$ are used in the inductive step. We also need the following property for the construction.

(P3) No leaf of any of the trees $T^i_{\overline{c}}$ or U^i_κ is a child of the root.

In the following section, we will describe the trees $T^i_{\overline{c}}$ and U^i_κ of height i and prove (P1) and (P2). Condition (P3) will be obvious from the construction. The subsequent section is then devoted to prove the effective automaticity of these trees.

4.2.1 Construction of trees We start with a few definitions concerning forests: Let H_1 and H_2 be two forests. The forest H_1^ω is the disjoint union of countably many copies of H_1. Formally, if $H_1 = (V; \leq)$, then $H_1^\omega = (V \times \mathbb{N}; \leq')$ with $(v, i) \leq' (w, j)$ if and only if $v \leq w$ and $i = j$. We write $H_1 \sim H_2$ for $H_1^\omega \cong H_2^\omega$. Thus, $H_1 \sim H_2$ if and only if they are formed, up to isomorphism, by the same set of trees (i.e., any tree is isomorphic to some connected component of H_1 if and only if it is isomorphic to some connected component of H_2). If H is a forest and r does not belong to the domain of H, then we denote with $r \circ H$ the tree that results from adding r to H as new least element. The construction in the following two Sections 4.2.1.1 and 4.2.1.2 is similar to a construction from [14] for levels of the hyperarithmetical hierarchy.

4.2.1.1 Induction base: construction of $T^2_{\overline{c}}$ and U^2_κ For notational simplicity, we write k for $1 + 2(n-2)$. Hence, P_2 is a k-ary predicate from Π^0_1. By Matiyasevich's theorem, we find two non-zero polynomials $p_1(x_1, \ldots, x_\ell), p_2(x_1, \ldots, x_\ell) \in \mathbb{N}[\overline{x}]$, $\ell > k$, such that for any $\overline{c} \in \mathbb{N}_+^k$:

$$P_2(\overline{c}) \text{ holds} \iff \forall \overline{x} \in \mathbb{N}_+^{\ell-k} : p_1(\overline{c}, \overline{x}) \neq p_2(\overline{c}, \overline{x}) . \tag{10}$$

For two numbers $m, n \in \mathbb{N}_+$, let $T[m, n]$ denote the tree of height 1 with exactly $C(m, n)$ leaves, where C is the injective polynomial function from (4). Then define the following forests:

$$\begin{aligned}
H^2 &= \biguplus\{T[m, n] \mid m, n \in \mathbb{N}_+, m \neq n\} , \\
H^2_{\overline{c}} &= H^2 \uplus \biguplus\{T[p_1(\overline{c}, \overline{x}) + x_{\ell+1}, p_2(\overline{c}, \overline{x}) + x_{\ell+1}] \mid \overline{x} \in \mathbb{N}_+^{\ell-k}, x_{\ell+1} \in \mathbb{N}_+\} \text{ and} \\
J^2_\kappa &= H^2 \uplus \biguplus\{T[x, x] \mid x \in \mathbb{N}_+, x > \kappa\} \quad \text{for } \kappa \in \mathbb{N}_+ \cup \{\omega\} .
\end{aligned}$$

Note that $J^2_\omega = H^2$. Moreover, the forests J^2_κ ($\kappa \in \mathbb{N}_+ \cup \{\omega\}$) are pairwise non-isomorphic, since C is injective.

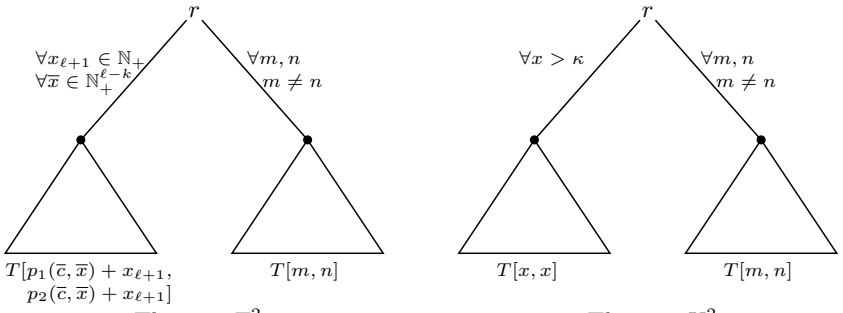
The trees $T^2_{\overline{c}}$ and U^2_κ, resp., are obtained from $H^2_{\overline{c}}$ and J^2_κ, resp., by taking countably many copies and adding a root (see Figure 2):

$$T^2_{\overline{c}} = r \circ (H^2_{\overline{c}})^\omega \text{ and } U^2_\kappa = r \circ (J^2_\kappa)^\omega . \tag{11}$$

The following lemma (stating (P1) for the Π^0_1-predicate P_2, (i.e., for $i = 2$) is proved in a similar way as Theorem 9.

Lemma 27. *For any $\overline{c} \in \mathbb{N}_+^k$, we have*

$$P_2(\overline{c}) \textit{ holds} \iff H^2_{\overline{c}} \sim J^2_\omega \iff T^2_{\overline{c}} \cong U^2_\omega .$$



The tree $T^2_{\overline{c}}$ The tree U^2_κ

Fig. 2. The tree $T^2_{\overline{c}}$ and U^2_κ

Proof. By (11), it suffices to show the first equivalence. First, assume that $P_2(\overline{c})$ holds. We have to prove that the forests $H^2_{\overline{c}}$ and $J^2_\omega = H^2$ contain the same trees (up to isomorphism). Clearly, every tree from H^2 is contained in $H^2_{\overline{c}}$. For the other direction, let $\overline{x} \in \mathbb{N}_+^{\ell-k}$ and $x_{\ell+1} \in \mathbb{N}_+$. Then the tree $T[p_1(\overline{c}, \overline{x}) + x_{\ell+1}, p_2(\overline{c}, \overline{x}) + x_{\ell+1}]$ occurs in $H^2_{\overline{c}}$. Since $P_2(\overline{c})$ holds, we have $p_1(\overline{c}, \overline{x}) \neq p_2(\overline{c}, \overline{x})$ by (10) and therefore $p_1(\overline{c}, \overline{x}) + x_{\ell+1} \neq p_2(\overline{c}, \overline{x}) + x_{\ell+1}$. Hence this tree also occurs in H^2.

Conversely suppose $H^2_{\overline{c}} \sim H^2$ and let $\overline{x} \in \mathbb{N}_+^{\ell-k}$. Then the tree $T[p_1(\overline{c}, \overline{x}) + 1, p_2(\overline{c}, \overline{x}) + 1]$ occurs in $H^2_{\overline{c}}$ and therefore in H^2. Hence $p_1(\overline{c}, \overline{x}) \neq p_2(\overline{c}, \overline{x})$. Since $\overline{x}$ was chosen arbitrarily, this implies $P_2(\overline{c})$. □

Lemma 28. *For every $\overline{c} \in \mathbb{N}_+^k$ there exists $\kappa \in \mathbb{N}_+ \cup \{\omega\}$ such that $T^2_{\overline{c}} \cong U^2_\kappa$.*

Proof. It suffices to show $H^2_{\overline{c}} \cong J^2_\kappa$ for some $\kappa \in \mathbb{N}_+ \cup \{\omega\}$. This holds, because if $H^2_{\overline{c}}$ contains a tree of the form $T[m, m]$ for some m (necessarily $m \geq 2$), then it contains all trees $T[x, x]$ for $x \geq m$. □

Lemma 27 and 28 imply the following lemma, which states (P2) for the Π^0_1-predicate P_2, i.e., for $i = 2$:

Lemma 29. *For any $\overline{c} \in \mathbb{N}_+^k$, we have:*

$$P_2(\overline{c}) \text{ does not hold} \iff \exists m \in \mathbb{N}_+ : T^2_{\overline{c}} \cong U^2_m \,.$$

This finishes the construction of the trees $T^2_{\overline{c}}$ and U^2_κ for $\kappa \in \mathbb{N}_+ \cup \{\omega\}$, and the verification of properties (P1) and (P2). Clearly, also (P3) holds for $T^2_{\overline{c}}$ and U^2_κ (all maximal paths have length 2).

4.2.1.2 Induction step: construction of $T^{i+1}_{\overline{c}}$ and U^{i+1}_κ For notational simplicity, we write again k for $1 + 2(n - i - 1)$ such that P_{i+1} is a k-ary predicate and P_i a $(k + 2)$-ary one.

We now apply the induction hypothesis. For any $\overline{c} \in \mathbb{N}_+^k$, $x, y \in \mathbb{N}_+$, and $\kappa \in \mathbb{N}_+ \cup \{\omega\}$, let $T^i_{\overline{c}xy}$ and U^i_κ be trees of height i such that:

- $P_i(\overline{c}, x, y)$ holds if and only if $T^i_{\overline{c}xy} \cong U^i_\omega$.
- $P_i(\overline{c}, x, y)$ does not hold if and only if $T^i_{\overline{c}xy} \cong U^i_m$ for some $m \in \mathbb{N}_+$.

Recall that, by point (i) of Lemma 26, the predicate $P_{i+1}(\overline{x})$ is logically equivalence to $\forall x_{n-i}\, \exists y_{n-i} : P_i(\overline{x}, x_{n-i}, y_{n-i})$. In a first step, we build the trees $T'_{\overline{c}xy}$ and $U'_{\kappa,x}$ ($x \in \mathbb{N}_+$) from $T^i_{\overline{c}xy}$ and U^i_κ, resp., by adding x leaves as children of the root. This ensures

$$T'_{\overline{c}xy} \cong T'_{\overline{c}x'y'} \iff x = x' \wedge T^i_{\overline{c}xy} \cong T^i_{\overline{c}x'y'} \text{ and} \tag{12}$$

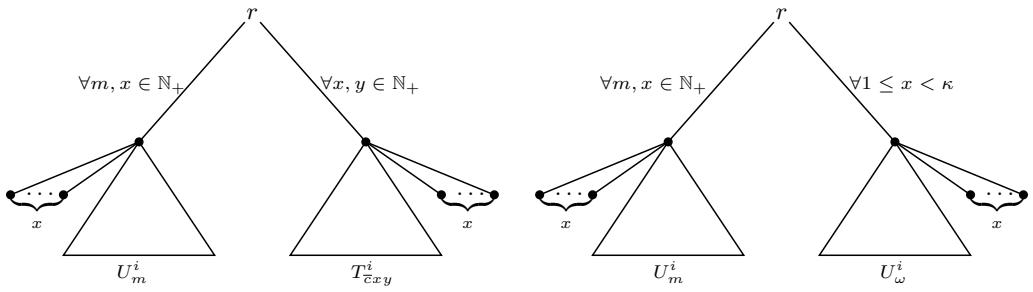
$$T'_{\overline{c}xy} \cong U'_{\kappa,x'} \iff x = x' \wedge T^i_{\overline{c}xy} \cong U^i_\kappa \,, \tag{13}$$

since, by property (P3), no leaf of any of the trees $T^i_{\overline{c}xy}$ or U^i_κ is a child of the root. Next, we collect these trees into forests as follows:

$$H^{i+1} = \biguplus \{U'_{m,x} \mid m, x \in \mathbb{N}_+\} \,,$$

$$H^{i+1}_{\overline{c}} = H^{i+1} \uplus \biguplus \{T'_{\overline{c}xy} \mid x, y \in \mathbb{N}_+\} \,, \text{ and}$$

$$J^{i+1}_\kappa = H^{i+1} \uplus \biguplus \{U'_{\omega,x} \mid 1 \leq x < \kappa\} \text{ for } \kappa \in \mathbb{N}_+ \cup \{\omega\}.$$



The tree $T^{i+1}_{\overline{c}}$ The tree U^{i+1}_{κ}

Fig. 3. The tree $T^{i+1}_{\overline{c}}$ and U^{i+1}_{κ}

The trees $T^{i+1}_{\overline{c}}$ and U^{i+1}_{κ}, resp., are then obtained from the forests $H^{i+1}_{\overline{c}}$ and J^{i+1}_{κ}, resp., by taking countably many copies and adding a root (see Figure 3):

$$T^{i+1}_{\overline{c}} = r \circ (H^{i+1}_{\overline{c}})^{\omega} \quad \text{and} \quad U^{i+1}_{\kappa} = r \circ (J^{i+1}_{\kappa})^{\omega} \,. \tag{14}$$

Note that the height of any of these trees is one more than the height of the forests defining them and therefore at most $i+1$. Since none of the connected components of the forests $H^{i+1}_{\overline{c}}$ and J^{i+1}_{κ} is a singleton, none of the trees in (14) has a leaf that is a child of the root and therefore (P3) holds.

The following lemma shows (P1) for the $\Pi^0_{2i-1}\ldots$-predicate P_{i+1}.

Lemma 30. *For any $\overline{c} \in \mathbb{N}_+^k$, we have*

$$P_{i+1}(\overline{c}) \text{ holds} \iff H^{i+1}_{\overline{c}} \sim J^{i+1}_{\omega} \iff T^{i+1}_{\overline{c}} \cong U^{i+1}_{\omega} \,.$$

Proof. Again, we only have to prove the first equivalence.

First assume $H^{i+1}_{\overline{c}} \sim J^{i+1}_{\omega}$ and let $x \geq 1$ be arbitrary. We have to exhibit some $y \geq 1$ such that $P_i(\overline{c}, x, y)$ holds. Note that $U'_{\omega,x}$ belongs to J^{i+1}_{ω} and therefore to $H^{i+1}_{\overline{c}}$. Since $U'_{\omega,x} \not\cong U'_{m,x'}$ for any $m, x, x' \in \mathbb{N}_+$, this implies the existence of $x', y' \geq 1$ with $T'_{\overline{c}x'y'} \cong U'_{\omega,x}$. By (13), this is equivalent with $x = x'$ and $T^i_{\overline{c}xy'} \cong U^i_{\omega}$. Now the induction hypothesis implies that $P_i(\overline{c}, x, y')$ holds. Since $x \geq 1$ was chosen arbitrarily, we can deduce $P_{i+1}(\overline{c})$.

Conversely suppose $P_{i+1}(\overline{c})$. Let T belong to $H^{i+1}_{\overline{c}}$. By the induction hypothesis, it is one of the trees $U'_{\kappa,x}$ for some $x \in \mathbb{N}_+$, $\kappa \in \mathbb{N}_+ \cup \{\omega\}$. In any case, it also belongs to J^{i+1}_{ω}. Hence it remains to show that any tree of the form $U'_{\omega,x}$ belongs to $H^{i+1}_{\overline{c}}$. So let $x \in \mathbb{N}_+$. Then, by $P_{i+1}(\overline{c})$, there exists $y \in \mathbb{N}_+$ with $P_i(\overline{c}, x, y)$. By the induction hypothesis, we have $T^i_{\overline{c}xy} \cong U^i_{\omega}$ and therefore $T'_{\overline{c}xy} \cong U'_{\omega,x}$ (which belongs to $H^{i+1}_{\overline{c}}$ by the very definition). □

The following lemma will be used to show (P2) for the $\Pi^0_{2i-1}\ldots$-predicate P_{i+1}.

Lemma 31. *For every $\overline{c} \in \mathbb{N}_+^k$, there exists $\kappa \in \mathbb{N}_+ \cup \{\omega\}$ such that $T^{i+1}_{\overline{c}} \cong U^{i+1}_{\kappa}$.*

Proof. It suffices to prove that $H^{i+1}_{\overline{c}} \sim J^{i+1}_{\kappa}$ for some $\kappa \in \mathbb{N}_+ \cup \{\omega\}$. Choose κ as the smallest value in $\mathbb{N}_+ \cup \{\omega\}$ such that

$$\forall x \geq \kappa \, \forall y : \neg P_i(\overline{c}, x, y)$$

holds. By property (ii) from Lemma 26 for P_i, we get

$$\forall 1 \leq x < \kappa \, \exists y : P_i(\overline{c}, x, y) \,.$$

By the induction hypothesis, we get

$$\forall x \geq \kappa \, \forall y : T'_{\overline{c}xy} \not\cong U'_{\omega,x} \;\text{ and }\; \forall 1 \leq x < \kappa \, \exists y : T'_{\overline{c}xy} \cong U'_{\omega,x} \,.$$

It follows that $H^{i+1}_{\overline{c}}$ contains, apart from the trees in $H^{i+1} = \biguplus\{U'_{m,x} \mid m, x \in \mathbb{N}_+\}$, exactly the trees from $\biguplus\{U'_{\omega,x} \mid 1 \leq x < \kappa\}$. Hence, $H^{i+1}_{\overline{c}} \sim J^{i+1}_{\kappa}$. □

Lemma 30 and 31 immediately imply (P2) for the $\Pi^0_{2i-1}\ldots$-predicate P_{i+1}:

Lemma 32. *For any $\overline{c} \in \mathbb{N}_+^k$, we have*

$$P_{i+1}(\overline{c}) \text{ does not hold} \iff \exists m \in \mathbb{N}_+ : T^{i+1}_{\overline{c}} \cong U^{i+1}_m .$$

In summary, we obtained the following:

Proposition 33. *For the Π^0_{2n-3}-predicate $P_n(x_0)$ we have for all $c \in \mathbb{N}_+$:*

$$P_n(c) \text{ holds} \iff T^n_c \cong U^n_\omega .$$

To infer the Π^0_{2n-3}-hardness of the isomorphism problems for $\mathcal{T}_n$ and $\mathcal{T}^{\mathrm{er}}_{n+1}$ from this proposition, it remains to be shown that the trees T^n_c and U^n_ω are effectively automatic – this is the topic of the next section.

4.2.2 Automaticity Automatic presentations of the trees $T^i_{\overline{c}}$ and U^i_m for $m \in \mathbb{N}_+ \cup \{\omega\}$ will be constructed inductively. Note that the construction of $T^{i+1}_{\overline{c}}$ involves all the trees $T^i_{\overline{c},x,y}$ for $x, y \in \mathbb{N}_+$. Hence we need *one single automatic presentation* for the forest consisting of all these trees. Therefore, we will deal with forests. To move from one forest to the next, we will always proceed as follows: add a set of new minimal elements below some of the old roots *which results in a partial order* (or poset) and not necessarily in a forest. The next forest will then be the unfolding of this poset.

A *path* in a poset $D = (V; \leq)$ is a maximal linearly ordered subset of V. The *height* of D is the maximal cardinality of a path. We only consider posets of finite height; thus all paths are finite linear orders. A poset $D = (V; \leq)$ of finite height can be unfolded into a forest unfold(D) (of the same height) in the usual way: Nodes of unfold(D) are nonempty prefixes of paths and a prefix p_1 is smaller than a prefix p_2, if p_1 is a proper initial segment of p_2. For a node $v \in V$ of D, we define the tree unfold(D, v) as the unfolding of the restriction of D to the set $\{x \in V \mid v \leq x\}$. We need the following lemma.

Lemma 34. *From $k \in \mathbb{N}$ and an automatic poset $D = (V; E)$ of height at most k, one can construct effectively an automatic presentation $\mathcal{P}$ with $\mathcal{S}(\mathcal{P}) \cong \mathrm{unfold}(D)$.*

Proof. The universe for our automatic copy of unfold(D) is the set P of all convolutions $v_1 \otimes v_2 \otimes \cdots \otimes v_m$, where $(v_1, v_2, \ldots, v_m)$ is a nonempty prefix of some path. Since D has height at most k, we have $m \leq k$. Since the order relation of D is automatic and since the set of all minimal elements of D is first-order definable and hence regular, P is indeed a regular set. Moreover, the order relation of unfold(D) becomes clearly FA recognizable on P. □

For $2 \leq i \leq n$, let us consider the following forest:

$$F_i = \biguplus \{T^i_{\overline{c}} \mid \overline{c} \in \mathbb{N}_+^{1+2(n-i)}\} \uplus \biguplus \{U^i_\kappa \mid \kappa \in \mathbb{N}_+ \cup \{\omega\}\} .$$

Let us take symbols a and b. For a tuple $\overline{c} \in \mathbb{N}_+^m$, recall the definition of the word $a^{\overline{c}} \in (\Sigma^a_m)^*$ from (2). Technically, this section proves by induction over i the following statement:

Proposition 35. *For $2 \leq i \leq n$, one can compute an automatic copy $\mathcal{F}_i$ of F_i and such that there exists an isomorphism $f_i : F_i \to \mathcal{F}_i$ that maps*

- *the root of the tree $T^i_{\overline{c}}$ to $a^{\overline{c}}$ (for all $\overline{c} \in \mathbb{N}_+^{1+2(n-i)}$),*
- *the root of the tree U^i_ω to ε, and*
- *the root of the tree U^i_m to b^m (for all $m \in \mathbb{N}_+$).*

This will give the desired result since T^n_c is then isomorphic to the connected component of $\mathcal{F}_n$ that contains the word a^c (and similarly for U^n_κ). Note that this connected component is automatic by Theorem 4 since it consists of all nodes above a^c. Moreover, an automatic presentation for the connected component containing a^c can be computed from c.

By Lemma 34, it suffices to construct an automatic poset $\mathcal{D}_i$ such that there is an isomorphism $h : \mathrm{unfold}(\mathcal{D}_i) \to \mathcal{F}_i$ that is the identity on the set of minimal elements of $\mathcal{D}_i$.

4.2.2.1 Induction base: the automatic poset $\mathcal{D}_2$ Recall the definitions of the language $\otimes_k(L)$ from (3). Moreover, recall the definition of the tree $T[k_1, k_2]$ for $k_1, k_2 \in \mathbb{N}_+$ from Section 4.2.1.1.

Lemma 36. *From $\ell \in \mathbb{N}_+$, $q_1, q_2 \in \mathbb{N}[x_1, \ldots, x_\ell]$, and a symbol a, one can compute an automatic forest of height 1 over an alphabet $\Sigma_\ell^a \uplus \Gamma$ such that*

- *the set of roots is $\otimes_\ell(a^+)$,*
- *the leaves are words from Γ^+, and*
- *the tree rooted at $a^{\overline{e}}$ is isomorphic to $T[q_1(\overline{e}), q_2(\overline{e})]$.*

Proof. Set $p(x_1, \ldots, x_\ell) = C(q_1(x_1, \ldots, x_\ell), q_2(x_1, \ldots, x_\ell))$, where C is the injective polynomial function from (4). From Lemma 7, we obtain a finite automaton $\mathcal{A}$ accepting $\otimes_k(a^+)$ such that $p(\overline{e}) = |\mathsf{Run}(\mathcal{A}, a^{\overline{e}})|$ for all $\overline{e} \in \mathbb{N}_+^\ell$. Then let

$$L[q_1, q_2] = \otimes_\ell(a^+) \cup \mathsf{Run}(\mathcal{A}) \text{ and}$$
$$E[q_1, q_2] = \{(u, v) \mid u \in \otimes_\ell(a^+), v \in \mathsf{Run}(\mathcal{A}, u)\}.$$

Then $L[q_1, q_2]$ is regular and $E[q_1, q_2]$ is FA recognizable, i.e., the pair $(L[q_1, q_2]; E[q_1, q_2] \cup \mathrm{id}_{L[q_1, q_2]})$ is an automatic graph. It is actually a forest of height 1; the words from $\otimes_\ell(a^+)$ form the roots, and the tree rooted at $a^{\overline{e}}$ has precisely $p(\overline{e})$ leaves, i.e., it is isomorphic to $T[q_1(\overline{e}), q_2(\overline{e})]$. □

From now on, we use the notations from Section 4.2.1.1. Using Lemma 36, we can compute automatic forests $\mathcal{G}_1$ and $\mathcal{G}_2$ over alphabets $\Sigma_{\ell+1}^a \uplus \Gamma_1$ and $\Sigma_2^b \uplus \Gamma_2$, respectively, such that

(a) the roots of $\mathcal{G}_1$ are the words from $\otimes_{\ell+1}(a^+)$,
(b) the roots of $\mathcal{G}_2$ are the words from $\otimes_2(b^+)$,
(c) the leaves of $\mathcal{G}_i$ are words from Γ_i^+ $(i \in \{1, 2\})$,
(d) $\mathcal{G}_1(a^{\overline{e}e_{\ell+1}}) \cong T[p_1(\overline{e}) + e_{\ell+1}, p_2(\overline{e}) + e_{\ell+1}]$ for $\overline{e} \in \mathbb{N}_+^\ell$, $e_{\ell+1} \in \mathbb{N}_+$, and
(e) $\mathcal{G}_1(b^{e_1 e_2}) \cong T[e_1, e_2]$ for $e_1, e_2 \in \mathbb{N}_+$.

We can assume that the alphabets Γ_1, Γ_2, $\Sigma_{\ell+1}^a$, and Σ_2^b are mutually disjoint. Let $\mathcal{F} = (V_\mathcal{F}; \leq_\mathcal{F})$ be the disjoint union of $\mathcal{G}_1$ and $\mathcal{G}_2$; it is effectively automatic.

The universe of the automatic poset $\mathcal{D}_2$ is the regular language

$$V_2 \;=\; \otimes_k(a^+) \cup b^* \cup (\$^* \otimes V_\mathcal{F}),$$

where \$ is a new symbol. The order relation $\leq_2$ of $\mathcal{D}_2$ is the least partial order that satisfies the following (for a set A, $a \leq A$ means that $a \leq b$ for all $b \in A$):

- For $u, v \in V_\mathcal{F}$ and $m, n \in \mathbb{N}$, $\$^m \otimes u \leq_2 \$^n \otimes v$ if and only if $m = n$ and $u \leq_\mathcal{F} v$. This produces $\aleph_0$ many copies of $\mathcal{F}$.
- $a^{\overline{c}} \leq_2 \$^* \otimes (\{a^{\overline{c}\,\overline{x}} \mid \overline{x} \in \mathbb{N}_+^{\ell-k+1}\} \cup \{b^{e_1 e_2} \mid e_1 \neq e_2\})$. By point (d) and (e) above, this means that the tree unfold$(\mathcal{D}_2, a^{\overline{c}})$ has $\aleph_0$ many subtrees isomorphic to $T[p_1(\overline{c}\,\overline{x}) + x_{\ell+1}, p_2(\overline{c}\,\overline{x}) + x_{\ell+1}]$ for $\overline{x} \in \mathbb{N}_+^{\ell-k}$, $x_{\ell+1} \in \mathbb{N}_+$ and $T[e_1, e_2]$ for $e_1, e_2 \in \mathbb{N}_+$, $e_1 \neq e_2$. Hence, unfold$(\mathcal{D}_2, a^{\overline{c}}) \cong T_{\overline{c}}^2$.
- $\varepsilon \leq_2 \$^* \otimes \{b^{e_1 e_2} \mid e_1 \neq e_2\}$. By (e) above, this means that the tree unfold$(\mathcal{D}_2, \varepsilon)$ has $\aleph_0$ many subtrees isomorphic to $T[e_1, e_2]$ for $e_1, e_2 \in \mathbb{N}_+$, $e_1 \neq e_2$. Hence, unfold$(\mathcal{D}_2, \varepsilon) \cong U_\omega^2$.
- $b^m \leq_2 \$^* \otimes \{b^{e_1 e_2} \mid e_1 \neq e_2 \text{ or } e_1 = e_2 > m\}$ for all $m \in \mathbb{N}_+$. By (e) above, this means that the tree unfold$(\mathcal{D}_2, b^m)$ has $\aleph_0$ many subtrees isomorphic to $T[e_1, e_2]$ for all $e_1, e_2 \in \mathbb{N}_+$ with $e_1 \neq e_2$ or $e_1 = e_2 > m$. Hence, unfold$(\mathcal{D}_2, b^m) \cong U_m^2$.

Thus, unfold$(\mathcal{D}_2) \cong F_2$ and the roots are as required in Proposition 35, see Figure 4. Moreover, it is easy to come up with an automaton for the partial order $\leq_2$ defined above. Hence, $\mathcal{D}_2$ is automatic.

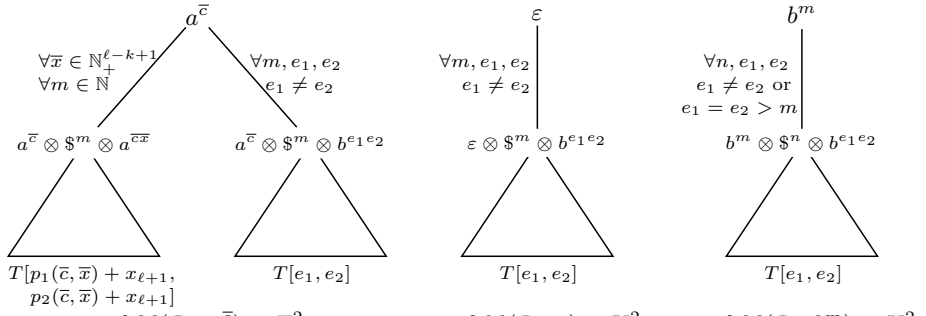


Fig. 4. Automatic presentation of $T^2_{\overline{c}}$ and U^2_κ

4.2.2.2 Induction step: the automatic poset $\mathcal{D}_{i+1}$ Suppose $\mathcal{D}_i = (V_i; \leq_i)$ is such that $\mathcal{F}_i = \text{unfold}(\mathcal{D}_i)$ is as described in Proposition 35.

We use the notations from Section 4.2.1.2. We first build another automatic poset $\mathcal{D}'$, whose unfolding consists of (copies of) all the trees $U'_{\kappa,x}$ ($\kappa \in \mathbb{N}_+ \cup \{\omega\}$, $x \in \mathbb{N}_+$) and $T'_{\overline{c}xy}$ ($\overline{c} \in \mathbb{N}_+^k$, $x, y \in \mathbb{N}_+$). Recall that the set of minimal elements of $\mathcal{D}_i$ is $\otimes_{k+2}(a^+) \cup b^* \subseteq V_i$. The universe of $\mathcal{D}'$ consists of the regular language

$$V' \;=\; (V \setminus b^*) \cup (\sharp^+ \otimes b^*) \cup \sharp_1^+ \sharp_2^* \,,$$

where $\sharp, \sharp_1$, and $\sharp_2$ are new symbols. The order relation $\leq'$ of $\mathcal{D}'$ is the least partial order containing $\leq_i$ (restricted to $V \setminus b^*$) that satisfies the following:

- $a^{\overline{c}xy} \leq' \{\sharp_1^j \sharp_2^{x-j} \mid 1 \leq j \leq x\}$ for all $\overline{c} \in \mathbb{N}_+^k, x, y \in \mathbb{N}_+$. This ensures that the subtree rooted at $a^{\overline{c}xy}$ gets x new leaves, which are children of the root. Hence unfold$(\mathcal{D}', a^{\overline{c}xy}) \cong T'_{\overline{c}xy}$.
- for $x \in \mathbb{N}_+$ and $m \in \mathbb{N}$, we have
  (i) $\sharp^x \otimes b^m \leq' \{h \in \mathcal{V}_i \mid b^m <_i h\}$ and
  (ii) $\sharp^x \otimes b^m \leq' \{\sharp_1^j \sharp_2^{x-j} \mid 1 \leq j \leq x\}$.
  This ensures that unfold$(\mathcal{D}', \sharp^x \otimes b^m) \cong U'_{m,x}$ in case $m \in \mathbb{N}_+$ and unfold$(\mathcal{D}', \sharp^x \otimes \varepsilon) \cong U'_{\omega,x}$.

In summary, $\mathcal{D}'$ is a poset, whose unfolding consists of (copies of) $U'_{\omega,x}$ rooted at $\sharp^x \otimes \varepsilon$, $U'_{m,x}$ ($m \in \mathbb{N}_+$) rooted at $\sharp^x \otimes b^m$, and $T'_{\overline{c}xy}$ rooted at $a^{\overline{c}xy}$.

From the automatic poset $\mathcal{D}'$, we now build in a final step the automatic poset $\mathcal{D}_{i+1}$. This is very similar to the constructions of $\mathcal{D}_2$ and $\mathcal{D}'$ above. The universe of $\mathcal{D}_{i+1}$ is the regular language

$$V_{i+1} \;=\; \otimes_k(a^+) \cup b^* \cup (\$^* \otimes V') \,.$$

The order relation $\leq_{i+1}$ of $\mathcal{D}_{i+1}$ is the least partial order satisfying:

- For $u, v \in V'$ and $m, n \in \mathbb{N}$, $\$^m \otimes u \leq_{i+1} \$^n \otimes v$ if and only if $m = n$ and $u \leq' v$. This generates $\aleph_0$ many copies of $\mathcal{D}'$.
- $a^{\overline{c}} \leq_{i+1} \$^* \otimes (\{a^{\overline{c}xy} \mid x, y \in \mathbb{N}_+\} \cup (\sharp^+ \otimes b^+))$. Hence, the tree unfold$(\mathcal{D}_{i+1}, a^{\overline{c}})$ has $\aleph_0$ many subtrees isomorphic to $T'_{\overline{c}xy}$ for $x, y \in \mathbb{N}_+$ and $U'_{m,x}$ for $x, m \in \mathbb{N}_+$. Thus, unfold$(\mathcal{D}_{i+1}, a^{\overline{c}}) \cong T^{i+1}_{\overline{c}}$.
- $\varepsilon \leq_{i+1} \$^* \otimes (\sharp^+ \otimes b^*)$. Hence, the tree unfold$(\mathcal{D}_{i+1}, \varepsilon)$ has $\aleph_0$ many subtrees isomorphic to $U'_{\kappa,x}$ for all $x \in \mathbb{N}_+$ and $\kappa \in \mathbb{N}_+ \cup \{\omega\}$. Thus, unfold$(\mathcal{D}_{i+1}, \varepsilon) \cong U^{i+1}_\omega$.
- $b^m \leq_{i+1} \$^* \otimes ((\sharp^+ \otimes b^+) \cup \{\sharp^x \otimes \varepsilon \mid 1 \leq x < m\})$ for $m \in \mathbb{N}_+$. This means that the tree unfold$(\mathcal{D}_{i+1}, b^m)$ has $\aleph_0$ many subtrees isomorphic to $U'_{m,x}$ for all $m, x \in \mathbb{N}_+$ and $U'_{\omega,x}$ for all $1 \leq x < m$. Hence, unfold$(\mathcal{D}_{i+1}, b^m) \cong U^{i+1}_m$.

See Figure 5, 6, and 7 for the overall construction. This finishes the proof of Proposition 35. Hence we obtain:

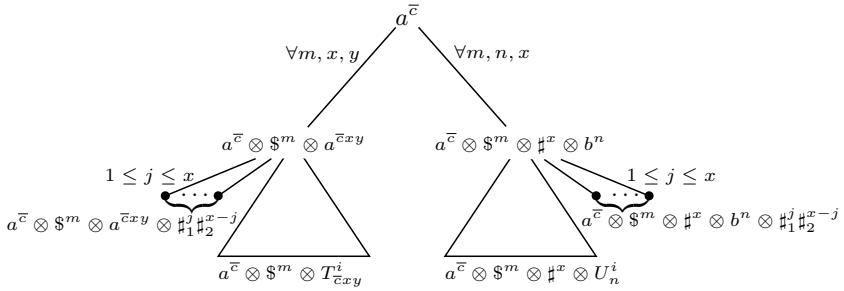


Fig. 5. Automatic presentation of $T^{i+1}_{\overline{c}}$

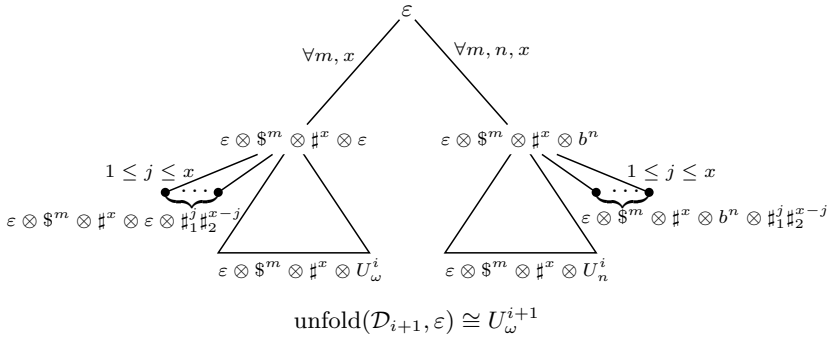


Fig. 6. Automatic presentation of U^{i+1}_ω

Theorem 37. *The following holds*

(a) For any $n \geq 2$, the isomorphism problem $\mathsf{Iso}(\mathcal{T}_n)$ for automatic trees of height at most n is Π^0_{2n-3}-complete.
(b) For any $n \geq 3$, the isomorphism problem $\mathsf{Iso}(\mathcal{T}^{\mathrm{er}}_n)$ for automatic trees of e-rank at most n is Π^0_{2n-5}-complete.
(c) The isomorphism problems for the following classes of automatic structures are recursively equivalent to $\mathsf{FOTh}(\mathbb{N}; +, \times)$: (i) automatic trees of finite height, (ii) well-founded automatic trees, (iii) automatic trees with only finitely many infinite branches.

Proof. We first prove (a). Containment in Π^0_{2n-3} was shown in Lemma 24. For the hardness, let $P_n \subseteq \mathbb{N}_+$ be any Π^0_{2n-3}-predicate and let $c \in \mathbb{N}_+$. Then, as above, we construct the automatic forest $\mathcal{F}_n$ of height n. The trees T^n_c and U^n_ω are first-order definable in $\mathcal{F}_n$ since they are (isomorphic to) the trees rooted at a^c and ε, resp. Hence these two trees are automatic. By Proposition 33, they are isomorphic if and only if $P_n(c)$ holds.

The upper bound in (b) is stated in Lemma 22. The lower bound follows as in (a) from the fact that $\mathsf{erank}(T^n_c) = \mathsf{erank}(U^n_\omega) = n + 1$ (one can embed into these trees ω^{n-1} but not ω^n).

The upper bound in (c) for the largest class (automatic trees with only finitely many infinite branches) is stated in Proposition 23. The lower bound for the smallest class (automatic trees of finite height) follows from our proof for (a), since the construction is uniform in the predicate P. □

Concerning the lower bound, we actually proved a slightly stronger statement: For every $n \geq 2$, there exists a fixed Π^0_{2n-3}-complete set $P_{2n-3} \subseteq \mathbb{N}_+$. If we apply our construction, we obtain a *fixed automatic forest* $\mathcal{F}_n$ of height n with the following properties: It is Π^0_{2n-3}-complete to determine, whether for a given $c \in \mathbb{N}_+$, the tree rooted at a^c in $\mathcal{F}_n$ is isomorphic to the tree rooted at ε in $\mathcal{F}_n$.

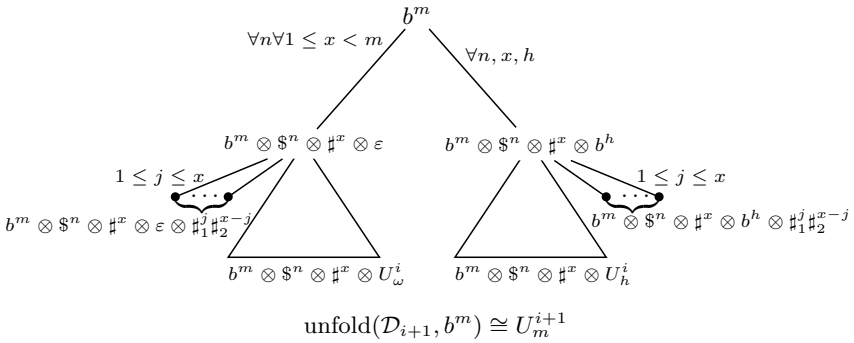


Fig. 7. Automatic presentation of U_m^{i+1}

4.3 Computable trees of finite height

In this section, we briefly discuss the isomorphism problem for computable trees of finite height. In an automatic tree, one can compute the root by Theorem 4 which is not the case for recursive trees. A similar remark concerns the edge relation E: in an automatic tree, it is FA recognizable, but in a computable tree, it need not be computable. But these two concepts (root and edge relation) are foundational for our proof of the upper bounds of the isomorphism problem. Therefore, here, we consider rooted successor trees, i.e., structures of the form $(V; E, r)$ where E is the edge relation of some tree $(V; \leq)$ with root r.

Theorem 38. *For every $n \geq 1$, the isomorphism problem for computable rooted successor trees of height at most n is Π^0_{2n}-complete.*

Proof. For the upper bound, let us first assume that $n = 1$. Two computable trees T_1 and T_2 of height 1 are isomorphic if and only if: for every $k \geq 0$, there exist at least k nodes in T_1 if and only if there exist at least k nodes in T_2. This is a Π^0_2-statement. For the inductive step, we can use arguments similar to those from the proof of Lemma 24.

For the lower bound, we first note that the isomorphism problem for computable rooted trees of height 1 is Π^0_2-complete. The problem whether a given recursively enumerable set is infinite is Π^0_2-complete [30]. For a given deterministic Turing-machine M, we construct a computable tree $T(M)$ of height 1 as follows: the set of leaves of $T(M)$ is the set of all accepting computations of M. We add a root to the tree and connect the root to all leaves. If $L(M)$ is infinite, then $T(M)$ is isomorphic to the height-1 tree with infinitely many leaves. If $L(M)$ is finite, then there exists $m \in \mathbb{N}$ such that $T(M)$ is isomorphic to the height-1 tree with m leaves. We can use this construction as the base case for our construction in Section 4.2.1.2. This yields the lower bound for all $n \geq 1$. □

4.4 Σ^1_1-hardness for trees

In this section, we prove that the isomorphism problem for the class of all automatic trees is hard (and hence complete) for Σ^1_1. In [23], the authors prove Σ^1_1-completeness for the isomorphism problem for automatic graphs. Their proof, as pointed out in [33], can be easily adapted to show the same lower bound for automatic *successor trees*. So, it is important to note that we work (as in all of this paper) with *order trees*, i.e., trees viewed as partial orders.

For a word $u = a_0 a_1 \cdots a_n \in \{0,1\}^*1$ with $a_i \in \{0,1\}$, let $\mathsf{num}(u) = \sum_{i=0}^{n} 2^i a_i$, i.e., u is the binary expansion of $\mathsf{num}(u)$ (least significant bit first).

Lemma 39. *There exists an automaton $\mathcal{A}_{\mathsf{num}}$ on the alphabet $\{0,1\}$ with $L_+(\mathcal{A}_{\mathsf{num}}) = \{0,1\}^*1$ such that $\mathsf{num}(w) = |\mathsf{Run}(\mathcal{A}_{\mathsf{num}}, w)|$ for all $w \in \{0,1\}^*1$.*

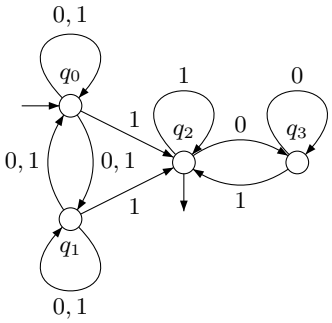


Fig. 8. The automaton $\mathcal{A}_{\mathsf{num}}$

Proof. Let $\mathcal{A}_{\mathsf{num}}$ be the automaton from Figure 8. Note that for each 1, the automaton can move from q_0 and q_1 resp. to the final state q_2, then the rest of the input is processed deterministically. If $\mathcal{A}_{\mathsf{num}}$ moves at input position k to the final state q_2, then there are 2^{k-1} possible runs until reaching input position k. Hence, if $p_1 < p_2 < \cdots < p_n$ are the 1-positions in an input $w \in \{0,1\}^*1$ ($p_1 \geq 1$, $p_n = |w|$) then $\mathcal{A}_{\mathsf{num}}$ has $\sum_{i=1}^{n} 2^{p_i - 1} = \mathsf{num}(w)$ accepting runs on w. □

The following lemma is analogous to Lemma 7. The only difference is that the first argument to the polynomial $p(x, \overline{y})$ is encoded in binary. To prove the lemma, we use the same construction as for Lemma 7 except the induction base $p(x) = x$ is handled by Lemma 39.

Lemma 40. *There exists an algorithm that, given a non-zero polynomial $p(x, \overline{y}) \in \mathbb{N}[x, \overline{y}]$ in $k+1$ variables, constructs an automaton $\mathcal{A}[p(x, \overline{y})]$ on the alphabet $(\{0,1\}^*1) \otimes \Sigma_k^a$ with $L_+(\mathcal{A}[p(x, \overline{y})]) = (\{0,1\}^*1) \otimes (\otimes_k(a^+))$ such that $p(\mathsf{num}(w), \overline{c}) = |\mathsf{Run}(\mathcal{A}[p(x, \overline{y})], w \otimes a^{\overline{c}})|$ for all $w \in \{0,1\}^*1$ and $\overline{c} \in \mathbb{N}_+^k$.*

The following lemma will be only used for the case that the set A is decidable. We state the lemma for arbitrary Σ_2^0 sets, since the proof is exactly the same.

Lemma 41. *There exist two trees U_0 and U_1 of height 3 ($U_0 \not\cong U_1$) with the following property: For a given index of a Σ_2^0-set $A \subseteq \{0,1\}^*1$ one can effectively construct an automatic forest $\mathcal{F}_A$ of height 3 such that:*

- *The set of roots of $\mathcal{F}_A$ is $\{0,1\}^*1$.*
- *For every $w \in \{0,1\}^*1$, $\mathcal{F}_A(w) \cong U_1$ if $w \in A$, and $\mathcal{F}_A(w) \cong U_0$ if $w \notin A$.*

Proof. Recall the definition of the trees U_κ^2 ($\kappa \in \mathbb{N}_+ \cup \{\omega\}$) from Section 4.2.1.1. We define the trees U_0 and U_1 as follows:

$$U_0 = r \circ \left(\biguplus \{U_m^2 \mid m \in \mathbb{N}_+\} \right)^{\aleph_0}$$

$$U_1 = r \circ \left(\biguplus \{U_\kappa^2 \mid \kappa \in \mathbb{N}_+ \cup \{\omega\}\} \right)^{\aleph_0}$$

Fix a Σ_2^0-set $A \subseteq \{0,1\}^*1$. Using the effectiveness of Matiyasevich's theorem, we can compute from an index of A two polynomials $p_1(x, y, \overline{z})$ and $p_2(x, y, \overline{z})$ such that

$$A = \{w \in \{0,1\}^*1 \mid \exists y \in \mathbb{N}_+ \; \forall \overline{z} \in \mathbb{N}_+^\ell : p_1(\mathsf{num}(w), y, \overline{z}) \neq p_2(\mathsf{num}(w), y, \overline{z})\}\,.$$

For the automatic forest $\mathcal{F}_A$, we repeat the construction from Section 4.2.2.1 with Lemma 40 instead of Lemma 7, i.e., the first argument to the polynomials p_1 and p_2 is binary encoded. We obtain an automatic forest $\mathcal{F}'$ that satisfies the following properties, which are analogous to Proposition 35:

– The set of roots of $\mathcal{F}'$ is $(\{0,1\}^*1 \otimes a^+) \cup b^*$
– $\mathcal{F}'(\varepsilon) \cong U^2_\omega$
– For all $m \in \mathbb{N}_+$, $\mathcal{F}'(b^m) \cong U^2_m$
– For all $w \in \{0,1\}^*1$ and $y \in \mathbb{N}_+$, $\mathcal{F}'(w \otimes a^y) \cong T^2_{\mathsf{num}(w),y}$.

The last property above implies together with Lemmas 27 and 29 the following properties for all $w \in \{0,1\}^*1$ and $y \in \mathbb{N}_+$:

$$\forall \overline{z} \in \mathbb{N}_+^\ell : p_1(\mathsf{num}(w), y, \overline{z}) \neq p_2(\mathsf{num}(w), y, \overline{z}) \text{ holds} \iff \mathcal{F}'(w \otimes a^y) \cong U^2_\omega$$
$$\exists \overline{z} \in \mathbb{N}_+^\ell : p_1(\mathsf{num}(w), y, \overline{z}) = p_2(\mathsf{num}(w), y, \overline{z}) \text{ holds} \iff \exists m \in \mathbb{N}_+ : \mathcal{F}'(w \otimes a^y) \cong U^2_m$$

Next, we construct an automatic poset $\mathcal{D}_A$ from the automatic forest $\mathcal{F}'$ as follows. Let $\mathcal{F}' = (L'; \leq')$. The universe of $\mathcal{D}_A$ is the set $\{0,1\}^*1 \cup (\$^* \otimes L')$. The order relation $\leq_A$ of $\mathcal{D}_A$ is the least partial order that satisfies the following:

– $\$^m \otimes u \leq_A \$^n \otimes v$ if and only if $m = n$ and $u \leq' v$ for $u, v \in L'$ and $m, n \in \mathbb{N}$. This generates $\aleph_0$ many copies of the forest $\mathcal{F}'$.
– $w \leq_A \$^* \otimes ((w \otimes a^+) \cup b^+)$ for all $w \in \{0,1\}^*1$.

Let $\mathcal{F}_A = \mathrm{unfold}(\mathcal{D}_A)$. Hence, for all $w \in \{0,1\}^*1$, we have

$$\mathcal{F}_A(w) \cong r \circ \left(\biguplus \{\mathcal{F}'(w \otimes a^y) \mid y \in \mathbb{N}_+\} \uplus \biguplus \{U^2_m \mid m \in \mathbb{N}_+\} \right)^{\aleph_0}.$$

From the definition of U_0 and U_1 and the properties of the forest $\mathcal{F}'$ it follows that for every $w \in \{0,1\}^*1$, $\mathcal{F}_A(w) \cong U_1$ if $w \in A$, and $\mathcal{F}_A(w) \cong U_0$ if $w \notin A$. □

Remark 42. It is possible to generalize the proof of Lemma 41 to arbitrary levels of the arithmetical hierarchy. Hence, for every $n \geq 2$, there exist trees U_0, U_1 of height $n+1$ such that from a given index of a Σ^0_n-set $A \subseteq \{0,1\}^*1$ one can compute an automatic forest $\mathcal{F}_A$ of height $n+1$ such that the following holds:

– The set of roots of $\mathcal{F}_A$ is $\{0,1\}^*1$.
– For every $w \in \{0,1\}^*1$, $\mathcal{F}_A(w) \cong U_1$ if $w \in A$, and $\mathcal{F}_A(w) \cong U_0$ if $w \notin A$.

This yields an alternative proof for the lower bound in Theorem 37(c). But only the (in fact more complicated) construction from Sections 4.2.1 and 4.2.2 yields the exact lower bounds stated in Theorem 37(a).

In [23], the authors prove Σ^1_1-completeness for the isomorphism problem for automatic graphs. Their proof, as pointed out in [33], can be easily adapted to show the same lower bound for (non-well-founded) automatic *successor trees*. So, it is important to note that the following theorem refers to *order trees*, i.e., trees viewed as partial orders.

Theorem 43. *There exists an order tree T such that the set of all automatic presentations $\mathcal{P}$ with $\mathcal{S}(\mathcal{P}) \cong T$ is Σ^1_1-complete. Hence the isomorphism problem for automatic order trees is Σ^1_1-complete.*

Proof. The proof combines the ideas from [23] and Lemma 41. In [23], the authors prove Σ^1_1-completeness for the isomorphism problem for automatic graphs by a reduction from the isomorphism problem for computable trees. For the latter problem, a *computable tree* is a prefix-closed and decidable subset $T \subseteq \mathbb{N}^*$. Such a tree is represented by an index for T. In the following, we construct for any computable tree $T \subseteq \mathbb{N}^*$ an automatic order tree $\mathsf{aut}(T)$ such that for two computable trees T_1, T_2, we have $T_1 \cong T_2$ if and only if $\mathsf{aut}(T_1) \cong \mathsf{aut}(T_2)$.

We start with the automatic presentation $(\{1\} \cup 1\{0,1\}^*1; \preceq)$ of the tree $(\mathbb{N}^*; \preceq)$, where in both structures $\preceq$ refers to the prefix relation. An isomorphism between these two trees is given by the computable mapping f with $f(n_1 n_2 \cdots n_k) = 10^{n_1}10^{n_2}1 \cdots 0^{n_k}1$ $(k \geq 0)$.

Let us fix a computable tree $T \subseteq \mathbb{N}^*$. Since T is computable and f is computable, the image $A = f(T) \subseteq \{1\} \cup 1\{0,1\}^*1 \subseteq \{0,1\}^*1$ is computable and hence a Σ^0_2-set. An index for A can be computed from an index for T. Now we apply Lemma 41 to the set A. We obtain an automatic forest $\mathcal{F}_A$ of height 3 (which can be constructed from an index for A) such that the following holds:

- The set of roots of $\mathcal{F}_A$ is $\{0,1\}^*1$.
- For every $w \in \{0,1\}^*1$, the subtree rooted in w is isomorphic to U_1 (resp. U_0) if $w \in A$ (resp. $w \notin A$).

In particular, any subtree rooted at $w \notin \{1\} \cup 1\{0,1\}^*1$ is isomorphic to U_0. Let $\mathcal{F}'_A$ be the subforest consisting of all trees of $\mathcal{F}_A$ rooted at some word from $\{1\} \cup 1\{0,1\}^*1$. Since this language is regular, $\mathcal{F}'_A$ is effectively automatic by Theorem 4.

Our automatic tree $\mathsf{aut}(T)$ results from the automatic forest $\mathcal{F}'_A$ by adding to the order relation the prefix relation on the set of $\mathcal{F}'_A$-roots $\{1\} \cup 1\{0,1\}^*1$. Intuitively, the tree $\mathsf{aut}(T)$ results from the tree $(\mathbb{N}^*; \preceq)$ by appending to each node $x \in \mathbb{N}^*$ a copy of the tree U_1 (resp. U_0) if $x \in T$ (resp. $x \notin T$). From this observation and $U_0 \not\cong U_1$, it follows that $T_1 \cong T_2 \iff \mathsf{aut}(T_1) \cong \mathsf{aut}(T_2)$ for all computable trees T_1 and T_2.

The first statement of the theorem follows from the fact that there exists a computable tree T whose computable copies form a Σ^1_1-complete set [13]. Our previous construction transforms this tree T into a fixed tree $\mathsf{aut}(T)$, whose set of automatic presentations is Σ^1_1-complete. □

5 Automatic Linear Orders

More details on linear orders can be found in [31]. We use ω to denote the linear order (type of) $(\mathbb{N}; \leq)$ of the natural numbers, ω^* for $(\{-n \mid n \in \mathbb{N}\}; \leq)$, ζ for $(\mathbb{Z}; \leq)$, and $\mathbf{n}$ for the finite linear order (type) of size n. Let $I = (D_I; \leq_I)$ be a linear order and, for $i \in D_I$, let $L_i = (D_i; \leq_i)$ be a linear order. The *sum* $\sum_{i \in I} L_i$ is the linear order $(\{(x,i) \mid i \in D_I, x \in D_i\}; \leq)$ where for all $i, j \in D_I$, $x \in D_i$, and $y \in D_j$,

$$(x,i) \leq (y,j) \iff i <_I j \vee (i = j \wedge x \leq_i y)\,.$$

We use $L_1 + L_2$ to denote $\sum_{i \in \mathbf{2}} L_i$ and $L_1 \cdot L_2$ to denote the sum $\sum_{i \in L_2} L^i_1$ where $L^i_1 = L_1$ for every $i \in L_2$. An *interval* or *convex subset* of a linear order $L = (D; \leq)$ is a subset $I \subseteq D$ such that $x, y \in I$ and $x < z < y$ imply $z \in I$. For $x, y \in D$ we write (x, y) for the interval $\{z \in D \mid x < z < y\}$.

Lemma 44. *If* $0 < 1$*, then* $(\{0,1\}^*1; \leq_{\mathsf{lex}}) \cong (\mathbb{Q}; \leq)$.

Proof. Let $u' \in \{0,1\}^*$ and $v \in \{0,1\}^*1$ such that $u = u'1 <_{\mathsf{lex}} v$. Then

$$u'01 <_{\mathsf{lex}} u <_{\mathsf{lex}} u0^{|v|}1 <_{\mathsf{lex}} v <_{\mathsf{lex}} v01\,.$$

Note that indeed $u0^{|v|}1 <_{\mathsf{lex}} v$: if u is a prefix of v, then $u0^{|v|}1$ and v differ (for the first time) at some position in the block $0^{|v|}$ where v carries 1. If u is no prefix of v, then $u0^{|v|}1$ and v differ (for the first time) at some position in u where u carries 0 and v carries 1.

Hence $(\{0,1\}^*1; \leq_{\mathsf{lex}})$ is countable, dense, and without endpoints. Thus, by Cantor's theorem (see [15]) it is isomorphic to $(\mathbb{Q}; \leq)$. □

The goal of this section is to prove that the isomorphism problem for automatic linear orders is Σ^1_1-complete. The general strategy of the proof is similar to the proof of Theorem 43 for automatic order trees. We will reduce the (Σ^1_1-complete) isomorphism problem for computable linear orders to the isomorphism problem for automatic linear orders. For this, we will need the following lemma:

Proposition 45. *From an index e of a computable linear order L, one can compute an index of a computable set $P(e) \subseteq \{0,1\}^*1$ whose complement is dense in $(\{0,1\}^*1; \leq_{\mathsf{lex}})$ such that*

$$L \cong L' \iff (\{0,1\}^*1; \leq_{\mathsf{lex}}, P(e)) \cong (\{0,1\}^*1; \leq_{\mathsf{lex}}, P(e'))$$

for all indices e and e' of computable linear orders L and L', resp.

Proof. Let e be an index of a computable linear order $L = (D; \leq)$. Then the product linear order $(\{0,1\}^*1; \leq_{\mathsf{lex}}) \cdot L$ is isomorphic to $(\{0,1\}^*1; \leq_{\mathsf{lex}}) \cong (\mathbb{Q}; \leq)$ (since it is countable, dense, and without endpoints) and an index for $(\{0,1\}^*1; \leq_{\mathsf{lex}}) \cdot L$ can be computed from e. Next, we use the well-known computable variant of Cantor's theorem: If L_1 and L_2 are computable copies of $(\mathbb{Q}; \leq)$, then there exists a computable isomorphism $f : L_1 \to L_2$ and an index for f can be computed from indices for L_1 and L_2 [6]. Applied to our situation, this means that from e we can compute an index for a computable isomorphism $f_e : (\{0,1\}^*1; \leq_{\mathsf{lex}}) \cdot L \to (\{0,1\}^*1; \leq_{\mathsf{lex}})$. Since the complement of $\{1\} \times D$ is dense in $(\{0,1\}^*1; \leq_{\mathsf{lex}}) \cdot L$, so is the complement of $P(e) = f_e(\{1\} \times D)$ in $(\{0,1\}^*1; \leq_{\mathsf{lex}})$. Since f_e is computable, the set $P(e)$ is computable too and an index for $P(e)$ can be computed from the index e.

The implication "$\Leftarrow$" follows since $(P(e); \leq_{\mathsf{lex}}) \cong L$ for any index e of the computable linear order L. Conversely, let $L = (D; \leq)$ and $L' = (D'; \leq')$ with index e and e', resp. Assume that $L \cong L'$. Then we have

$$\begin{aligned}(\{0,1\}^*1; \leq_{\mathsf{lex}}, P(e)) &\cong (\{0,1\}^*1 \times D; \sqsubseteq, \{1\} \times D) \\ &\cong (\{0,1\}^*1 \times D'; \sqsubseteq', \{1\} \times D') \text{ since } L \cong L' \\ &\cong (\{0,1\}^*1; \leq_{\mathsf{lex}}, P(e))\end{aligned}$$

where $\sqsubseteq$ is the order of $(\{0,1\}^*1; \leq_{\mathsf{lex}}) \cdot L$ and similarly for $\sqsubseteq'$. □

The following section will, from $P \subseteq \{0,1\}^*1$, construct a linear order $\mathsf{aut}(P)$ and prove that $(\{0,1\}^*1; \leq_{\mathsf{lex}}, P) \cong (\{0,1\}^*1; \leq_{\mathsf{lex}}, P')$ if and only if $\mathsf{aut}(P) \cong \mathsf{aut}(P')$ (assuming the complements of P and P' are dense). The subsequent section will then prove that $\mathsf{aut}(P)$ is effectively automatic for P computable.

5.1 Construction of linear orders

A key technique used in the construction is the shuffle sum of a class of linear orders. Let I be a countable set. A *dense I-coloring* of a linear order $L = (D; \leq)$ is a mapping $c : D \to I$ such that for all $x, z \in D$ with $x < z$ and all $i \in I$ there exists $y \in (x, z)$ with $c(y) = i$. Equivalently, $c^{-1}(i)$ is dense in $(D; \leq)$ for all $i \in I$.

Definition 46. *Let $\mathcal{L}$ be a countable set of linear orders and let $c : \mathbb{Q} \to \mathcal{L}$ be a dense $\mathcal{L}$-coloring of $\mathbb{Q}$. The shuffle sum of $\mathcal{L}$, denoted $\mathsf{Shuf}(\mathcal{L})$, is the linear order $\sum_{x \in (\mathbb{Q}; \leq)} c(x)$.*

Extending the classical back-and-forth construction of isomorphisms, one obtains that $(\mathbb{Q}; \leq, c_1) \cong (\mathbb{Q}; \leq, c_2)$ for any two dense I-colorings c_1 and c_2 of $\mathbb{Q}$. Hence, in the above definition, the isomorphism type of $\sum_{x \in (\mathbb{Q}; \leq)} c(x)$ does not depend on the choice of the dense $\mathcal{L}$-coloring c, see e.g. [31]. This implies that $\mathsf{Shuf}(\mathcal{L})$ is indeed uniquely defined.

For $m, n \in \mathbb{N}_+$, let $L[m,n]$ be the finite linear order with $C(m,n)$ elements (recall from (4) on page 9 that the polynomial function $C(x,y) = (x+y)^2 + 3x + y$ is injective). For $\kappa \in \mathbb{N}_+ \cup \{\omega\}$, we define the class of linear orders $\mathcal{L}'_\kappa$ and the linear order L'_κ as follows:

$$\mathcal{L}'_\kappa = \{L[m,n] \mid m, n \in \mathbb{N}_+, m \neq n \text{ or } m = n \geq \kappa\} \text{ and} \tag{15}$$

$$L'_\kappa = \mathsf{Shuf}(\mathcal{L}'_\kappa)\,. \tag{16}$$

Next, two linear orders M_0 and M_1 are given by

$$\begin{aligned}M_0 &= \mathsf{Shuf}\{L'_\kappa \mid 2 \leq \kappa < \omega\} \text{ and} \\ M_1 &= \mathsf{Shuf}\{L'_\kappa \mid 2 \leq \kappa \leq \omega\}\,.\end{aligned}$$

Finally, let $P \subseteq \{0,1\}^*1$. Then the linear order $\mathsf{aut}(P)$ is obtained from $(\{0,1\}^*1; \leq_{\mathsf{lex}}, P)$ by replacing every element of P by M_1 and every element of $\{0,1\}^*1 \setminus P$ by M_0:

$$\mathsf{aut}(P) = \sum_{x \in (\{0,1\}^*1; \leq_{\mathsf{lex}})} L_x^P \quad \text{with} \quad L_x^P = \begin{cases} M_1 & \text{if } x \in P \\ M_0 & \text{if } x \notin P \end{cases}$$

By the very definition, $(\{0,1\}^*1; \leq_{\mathsf{lex}}, P) \cong (\{0,1\}^*1; \leq_{\mathsf{lex}}, R)$ implies $\mathsf{aut}(P) \cong \mathsf{aut}(R)$. The following example shows that the converse is false in general.

Example 47. Let $P = \{1\}$ and let R be the open interval $(1, 11)$. Thus, $(\{0,1\}^*1; \leq_{\mathsf{lex}}, P) \ncong (\{0,1\}^*1; \leq_{\mathsf{lex}}, R)$. Since $\sum_{x \in ((1,11); \leq_{\mathsf{lex}})} L_x$ with $L_x = M_1$ for all $x \in (1, 11)$ is isomorphic to M_1, we get $\mathsf{aut}(P) \cong \mathsf{aut}(R)$.

Note that above the complement of P is dense, but the complement of R is not. The following lemmas prepare the proof that density of the complement is the only obstacle, see Proposition 52.

Lemma 48. *Let $\mathsf{Shuf}(\mathcal{L}) = \sum_{p \in (\mathbb{Q}; \leq)} c(p)$ be a shuffle sum. For all $x \in \mathsf{Shuf}(\mathcal{L})$ there exists $y > x$ such that the interval (x, y) contains an interval isomorphic to $\mathsf{Shuf}(\mathcal{L})$.*

Proof. Let $x = (q, a)$ with $q \in \mathbb{Q}$ and $a \in c(q)$. Choose an arbitrary $r \in \mathbb{Q}$ with $r > q$. A direct back-and-forth argument shows $((q, r); \leq, c) \cong (\mathbb{Q}; \leq, c)$ and therefore $\sum_{p \in ((q,r); \leq)} c(p) \cong \sum_{p \in (\mathbb{Q}; \leq)} c(p) = \mathsf{Shuf}(\mathcal{L})$, which implies the lemma. □

Lemma 49. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be countable sets of linear orders. Let $L = (D; \leq)$ be a linear order and I_1 and I_2 intervals of L with $I_1 \cong \mathsf{Shuf}(\mathcal{L}_1)$, $I_2 \cong \mathsf{Shuf}(\mathcal{L}_2)$, and $I_1 \cap I_2 \neq \emptyset$. Then $\mathsf{Shuf}(\mathcal{L}_1)$ is isomorphic to an interval of $\mathsf{Shuf}(\mathcal{L}_2)$ or vice versa.*

Proof. Let $x \in I_1 \cap I_2$. By Lemma 48 there exist $y_1 \in I_1$ and $y_2 \in I_2$ such that (x, y_i) contains an interval isomorphic to $\mathsf{Shuf}(\mathcal{L}_i)$ for $i \in \{1, 2\}$. W.l.o.g. assume that $y_1 \leq y_2$ in L. Then (x, y_1) is contained in $I_2 \cong \mathsf{Shuf}(\mathcal{L}_2)$, which proves the lemma. □

Lemma 50. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two countable sets of finite linear orders.*

(1) Any infinite interval of $\mathsf{Shuf}(\mathcal{L}_1)$ contains an interval isomorphic to $\mathsf{Shuf}(\mathcal{L}_1)$.
(2) If $\mathsf{Shuf}(\mathcal{L}_1)$ is isomorphic to some interval of $\mathsf{Shuf}(\mathcal{L}_2)$, then $\mathcal{L}_1 = \mathcal{L}_2$ (up to isomorphism).
(3) Let $L = (D; \leq)$ be a linear order and I_1 and I_2 intervals of L with $I_1 \cong \mathsf{Shuf}(\mathcal{L}_1)$, $I_2 \cong \mathsf{Shuf}(\mathcal{L}_2)$, and $I_1 \cap I_2 \neq \emptyset$. Then $\mathcal{L}_1 = \mathcal{L}_2$ (up to isomorphism).

Proof. For $i \in \{1, 2\}$, let $c_i : \mathbb{Q} \to \mathcal{L}_i$ be a dense $\mathcal{L}_i$-coloring. For claim (1), let I be some infinite interval in $\sum_{x \in (\mathbb{Q}; \leq)} c_1(x)$. Since $c_1(x)$ is finite for all $x \in \mathbb{Q}$, there are $x, y \in \mathbb{Q}$ such that $x < y$ and both $c_1(x)$ and $c_1(y)$ intersect I. Hence $\sum_{z \in ((x,y); \leq)} c_1(z)$ is an interval in I. A direct back-and-forth argument shows $((x, y); \leq, c_1) \cong (\mathbb{Q}; \leq, c_1)$ and therefore $\sum_{z \in ((x,y); \leq)} c_1(z) \cong \sum_{x \in (\mathbb{Q}; \leq)} c_1(x) \cong \mathsf{Shuf}(\mathcal{L}_1)$.

For claim (2), suppose $\sum_{x \in (\mathbb{Q}; \leq)} c_1(x)$ is isomorphic to some interval I of $\sum_{x \in (\mathbb{Q}; \leq)} c_2(x)$. Then there is an embedding $f : \sum_{x \in (\mathbb{Q}; \leq)} c_1(x) \to \sum_{x \in (\mathbb{Q}; \leq)} c_2(x)$ whose range I is convex.

First let $L \in \mathcal{L}_1$. Since c_1 is surjective, there exists $x \in \mathbb{Q}$ with $L = c_1(x)$. Let $a \in L$ and $(y, b) = f(x, a)$ with $y \in \mathbb{Q}$. The maximal finite interval of $\sum_{x \in (\mathbb{Q}; \leq)} c_1(x)$ containing (x, a) is isomorphic to $c_1(x) = L$. Since f is an embedding with convex range, the maximal finite interval of $\sum_{x \in (\mathbb{Q}; \leq)} c_2(x)$ containing $f(x, a)$ is isomorphic to L as well. Since it contains (y, b), it is at the same time isomorphic to $c_2(y)$. Hence, indeed, there exists $L' \in \mathcal{L}_2$ with $L \cong L'$.

For the converse implication, note that by (1), $I \cong \mathsf{Shuf}(\mathcal{L}_1)$ contains an interval isomorphic to $\mathsf{Shuf}(\mathcal{L}_2)$. By symmetry, we therefore obtain from the previous paragraph that, for any $L' \in \mathcal{L}_2$, there exists $L \in \mathcal{L}_1$ with $L \cong L'$. This proves claim (2).

Finally, claim (3) follows from claim (2) and Lemma 49. □

Lemma 51. *The linear order M_1 is not isomorphic to any interval of the linear order M_0 and vice versa.*

Proof. Let $M_i = \sum_{x \in (\mathbb{Q};\leq)} c_i(x)$ for $i \in \{1,2\}$, where $c_0 : \mathbb{Q} \to \{L'_\kappa \mid 2 \leq \kappa < \omega\}$ and $c_1 : \mathbb{Q} \to \{L'_\kappa \mid 2 \leq \kappa \leq \omega\}$ are dense colorings. We start with the following claim:

Claim: Let $i \in \{0,1\}$, f be an embedding of M_i into M_{1-i}, and $(x,a) \in M_i$, $(y,b) = f(x,a)$ (with $x, y \in \mathbb{Q}$, $a \in c_i(x)$, $b \in c_{1-i}(y)$). Then $c_i(x) \cong c_{1-i}(y)$.

To see this, note that (y,b) belongs to an interval isomorphic to $c_{1-i}(y)$ by definition. Moreover, since f is an embedding with convex range, it also belongs to an interval isomorphic to $c_i(x)$. The claim follows from Lemma 50(3).

Let us now prove the lemma and assume first, towards a contradiction, that f is an embedding of M_1 into M_0 with convex range. Let $x \in \mathbb{Q}$ such that $c_1(x) = L'_\omega$. Chose an element $a \in L'_\omega$ and consider $(x,a) \in M_0$. Let $(x',a') = f(x,a) \in M_0$ with $x' \in \mathbb{Q}$. By the above claim, we get $c_0(x') \cong c_1(x) = L'_\omega$, which contradicts the fact that no order isomorphic to L'_ω belongs to the range of c_0.

For the other case, suppose that f is an embedding of M_0 into M_1 with convex range I. Choose $x, y \in \mathbb{Q}$ with $x < y$ and $c_0(x) \ncong c_0(y)$ and let $a \in c_0(x)$, $b \in c_0(y)$. Let $(x',a') = f(x,a) \in I$ and $(y',b') = f(y,b) \in I$. Using our claim, we get $c_1(x') \cong c_0(x) \ncong c_0(y) \cong c_1(y')$. Since $(x,a) < (y,b)$ in M_0, we must have $(x',a') < (y',b')$ in M_1. Because of $c_1(x') \ncong c_1(y')$, we have $x' < y'$ in $\mathbb{Q}$. Take $z' \in \mathbb{Q}$ with $x' < z' < y'$ and $c' \in c_1(z') = L'_\omega$. Then $(z',c') \in I$, hence $(z,c) = f^{-1}(z',c')$ is defined. Our claim yields $c_0(z) \cong c_1(z') = L'_\omega$, which is a contradiction. □

Proposition 52. *Let the complements of $P, R \subseteq \{0,1\}^*1$ be dense in $(\{0,1\}^*1; \leq_{\mathsf{lex}})$. Then $\mathsf{aut}(P) \cong \mathsf{aut}(R)$ if and only if $(\{0,1\}^*1; \leq_{\mathsf{lex}}, P) \cong (\{0,1\}^*1; \leq_{\mathsf{lex}}, R)$.*

Proof. The if-direction is trivial since any isomorphism from $(\{0,1\}^*1; \leq_{\mathsf{lex}}, P)$ to $(\{0,1\}^*1; \leq_{\mathsf{lex}}, R)$ induces an isomorphism from $(P; \leq_{\mathsf{lex}})$ to $(R; \leq_{\mathsf{lex}})$. For the other implication, we show that the elements of P are in one-to-one correspondence with the maximal intervals in $\mathsf{aut}(P)$ of type M_1.

Let I be an interval of $\mathsf{aut}(P)$ with $I \cong M_1$ and suppose there is $x \in \{0,1\}^*1 \setminus P$ with $(\{x\} \times M_0) \cap I \neq \emptyset$. Hence an interval of type M_1 intersects an interval of type M_0. Lemma 49 implies that M_0 is isomorphic to an interval of type M_1 or vice versa, which leads to a contradiction by Lemma 51. Thus, $I \subseteq P \times M_1$.

Let $p \in P$. Then $\{p\} \times M_1$ is an interval in $\mathsf{aut}(P)$ of type M_1. Let $I \supsetneq \{p\} \times M_1$ be an interval of type M_1 properly larger than $\{p\} \times M_1$. Then there is $x \in \{0,1\}^*1$ with $p \neq x$ such that I contains an element of the form (x,u). Since the complement of P is dense in $(\{0,1\}^*1; \leq_{\mathsf{lex}})$, there is $y \in \{0,1\}^*1 \setminus P$ such that $p <_{\mathsf{lex}} y <_{\mathsf{lex}} x$ or $x <_{\mathsf{lex}} y <_{\mathsf{lex}} p$. Since I is an interval, we have $(\{y\} \times M_0) \cap I \neq \emptyset$, contradicting the above observation $I \subseteq P \times M_1$. Hence, the maximal intervals in $\mathsf{aut}(P)$ of type M_1 are precisely the intervals of the form $\{p\} \times M_1$. Since the corresponding statement holds for the maximal intervals in $\mathsf{aut}(R)$ of type M_1, any isomorphism from $\mathsf{aut}(P)$ to $\mathsf{aut}(R)$ induces an isomorphism from $(P; \leq_{\mathsf{lex}})$ to $(R; \leq_{\mathsf{lex}})$. □

Let e and e' be indices of computable linear orders L and L' and let $P(e)$ and $P(e')$ be the computable sets computed in Proposition 45. Since the complements of these sets are dense, we have $L \cong L'$ if and only if $\mathsf{aut}(P(e)) \cong \mathsf{aut}(P(e'))$. In the following section, we will prove that an automatic presentation of the linear order $\mathsf{aut}(P(e))$ can be computed from e.

5.2 Automaticity

In this section, let $P \subseteq \{0,1\}^*1$ be a Π^0_1-set with dense complement. We will effectively construct an automatic presentation of $\mathsf{aut}(P)$.

Recall that for $u \in \{0,1\}^*1$, $\mathsf{num}(u) \in \mathbb{N}_+$ is the unique natural number such that u is the binary expansion of $\mathsf{num}(u)$ (least significant bit first). Since $P \in \Pi^0_1$, the set $\{\mathsf{num}(u) \mid u \in P\}$

is the complement of a recursively enumerable set. Hence (from an index of P), one can compute $\ell \in \mathbb{N}_+$ and two polynomials $p_1, p_2 \in \mathbb{N}[x, y_1, \ldots, y_\ell]$ such that for all $u \in \{0,1\}^*1$:

$$u \in P \iff \forall \overline{n} \in \mathbb{N}_+^\ell : p_1(\mathsf{num}(u), \overline{n}) \neq p_2(\mathsf{num}(u), \overline{n})\,.$$

In the rest of this section, we fix the number ℓ and the polynomials p_1 and p_2. We define the two languages

$$D = (0^*1)^+ \text{ and } E = ((0^+1)^{\ell+1}\{a,b,c\})^+\,.$$

On the alphabet $\{0,1,a,b,c\}$ let us fix the order

$$0 < 1 < a < b < c,$$

which gives us a lexicographical order $\leq_{\mathsf{lex}}$ on $\{0,1,a,b,c\}^*$. For $u \in D$ let $\mathrm{col}_D(u) \in \mathbb{N}$ be the length of the last 0-block, i.e., $\mathrm{col}_D(u0^n1) = n$ for $u \in (0^*1)^*$. For $u \in ((0^+1)^{\ell+1}\{a,b,c\})^*$, $n_1, \ldots, n_{\ell+1} \in \mathbb{N}_+$, and $d \in \{a,b,c\}$ let $\mathrm{col}_E(u0^{n_1}10^{n_2}1\cdots 0^{n_{\ell+1}}1d) = (n_1, \ldots, n_{\ell+1}, d)$.

Lemma 53. *The following holds:*

- $(D; \leq_{\mathsf{lex}}) \cong (\mathbb{Q}; \leq)$ *and* col_D *is a dense* $\mathbb{N}$*-coloring of* $(D; \leq_{\mathsf{lex}})$.
- $(E; \leq_{\mathsf{lex}}) \cong (\mathbb{Q}; \leq)$ *and* col_E *is a dense* $(\mathbb{N}_+^{\ell+1} \times \{a,b,c\})$*-coloring of* $(E; \leq_{\mathsf{lex}})$.

Proof. For the first statement, let $u, v \in D$ with $u'1 = u <_{\mathsf{lex}} v$ and let $n \in \mathbb{N}$. Then

$$u'01 <_{\mathsf{lex}} u <_{\mathsf{lex}} u0^{|v|}10^n1 <_{\mathsf{lex}} v <_{\mathsf{lex}} v01\,.$$

Note that indeed $u0^{|v|}10^n1 <_{\mathsf{lex}} v$: if u is a prefix of v, then $u0^{|v|}10^n1$ and v differ (for the first time) at some position in the block $0^{|v|}$ where v carries 1. If u is no prefix of v, then $u0^{|v|}10^n1$ and v differ (for the first time) at some position in u where u carries 0 and v carries 1. Hence $(D; \leq_{\mathsf{lex}})$ is countable, dense and without endpoints and therefore isomorphic to $(\mathbb{Q}; \leq)$. Furthermore, $\mathrm{col}_D^{-1}(n)$ is dense for all $n \in \mathbb{N}$, i.e., col_D is indeed a dense $\mathbb{N}$-coloring.

For the second statement, let $u, v \in E$ with $u'1x_u = u <_{\mathsf{lex}} v = v'1x_v$ (where $x_u, x_v \in \{a,b,c\}$), $n_1, \ldots, n_{\ell+1} \in \mathbb{N}_+$, and $d \in \{a,b,c\}$. Then

$$u'01a <_{\mathsf{lex}} u <_{\mathsf{lex}} u(0^{|v|}1)^{\ell+1}a0^{n_1}10^{n_2}1\cdots 0^{n_{\ell+1}}1d <_{\mathsf{lex}} v <_{\mathsf{lex}} v(01)^{\ell+1}a\,.$$

Arguments analogous to those above show the claims regarding E and col_E. □

Lemma 54. *From a polynomial* $p \in \mathbb{N}[x, y, z_1, \ldots, z_{\ell+1}]$*, one can compute a nondeterministic finite automaton* $\mathcal{A}_p$ *with* $L(\mathcal{A}_p) = \{0,1\}^*1\#D\#E$ *that has precisely* $p(\mathsf{num}(u), \mathrm{col}_D(v), \overline{n})$ *accepting runs on* $u\#v\#w \in \{0,1\}^*1\#D\#E$ *with* $\mathrm{col}_E(w) = (\overline{n}, d)$ *for any* $d \in \{a,b,c\}$.

Proof. We proceed by induction on the construction of the polynomial p. First, consider the polynomial $p = x$ (for which the argument is very similar to the proof of Lemma 39). Let $\mathcal{A}_x$ be the automaton from Figure 9. Note that for each 1 preceeding any $\#, a, b, c$, the automaton can move from q_0 and q_1 resp. to q_2, then the rest of the input is processed deterministically. If $\mathcal{A}_x$ moves at input position k to the state q_2, then there are 2^{k-1} possible runs until reaching input position k. Hence, if $p_1 < p_2 < \cdots < p_n$ are the 1-positions in u ($p_1 \geq 1$, $p_n = |u|$) then $\mathcal{A}_x$ has $\sum_{i=1}^n 2^{p_i - 1} = \mathsf{num}(u)$ accepting runs on $u\#v\#w$.

Next consider the polynomial $p = y$ and let $\mathcal{A}_y$ be the automaton from Figure 10. Let $v \in D$. Since $0^{\mathrm{col}_D(v)}1$ is the longest suffix of v from 0^*1, there are precisely $\mathrm{col}_D(v)$ runs labeled v from q_1 to q_3. Since the rest of the automaton is deterministic, there are precisely $\mathrm{col}_D(v)$ many accepting runs on $u\#v\#w \in \{0,1\}^*1\#D\#E$.

Now consider the polynomial $p = z_i$ for some $1 \leq i \leq \ell + 1$. For $i = \ell$, the automaton $\mathcal{A}_{z_i}$ is depicted in Figure 11. Let $w \in E$ with $\mathrm{col}_E(w) = (n_1, \ldots, n_\ell, n_{\ell+1}, d)$. Then $0^{n_\ell}10^{n_{\ell+1}}1d$ is the longest suffix of w from $0^*10^*1\{a,b,c\}$. Hence there are precisely n_ℓ runs labeled w from q_0 to q_4. Since the rest of the word $u\#v\#w$ is processed deterministically, there are precisely n_ℓ many accepting runs on $u\#v\#w \in \{0,1\}^*1\#D\#E$.

The rest of the proof follows that of Lemma 7. □

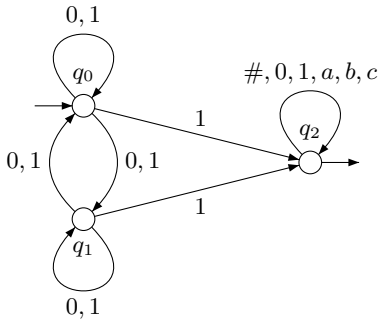


Fig. 9. The automaton $\mathcal{A}_x$

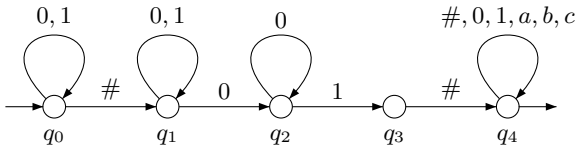


Fig. 10. The automaton $\mathcal{A}_y$

Using the above lemma, one can compute a nondeterministic finite automaton $\mathcal{A}$ with $L_+(\mathcal{A}) = \{0,1\}^*1\#D\#E$ such that for all $u \in \{0,1\}^*1$, $v \in D$, and $w \in E$ with $\mathrm{col}_E(w) = (\overline{n}, n_{\ell+1}, d)$ and $\overline{n} = (n_1, \ldots, n_\ell)$:

$$|\mathsf{Run}(\mathcal{A}, u\#v\#w)| = \begin{cases} C(p_1(\mathsf{num}(u), \overline{n}) + n_{\ell+1}, p_2(\mathsf{num}(u), \overline{n}) + n_{\ell+1}) & \text{if } d = a \text{ and } \mathrm{col}_D(v) = 0 \\ C(\mathrm{col}_D(v) + n_1, \mathrm{col}_D(v) + n_1) & \text{if } d = a \text{ and } \mathrm{col}_D(v) > 0 \\ C(n_1, n_1 + n_2) & \text{if } d = b \\ C(n_1 + n_2, n_1) & \text{if } d = c\,. \end{cases} \tag{17}$$

Recall that $\mathsf{Run}(\mathcal{A})$ is the set of accepting runs of $\mathcal{A}$, that, for $r \in \mathsf{Run}(\mathcal{A})$, $\mathrm{lab}(r)$ is the word accepted by r, and that $\mathsf{Run}(\mathcal{A}, u) = \{r \in \mathsf{Run}(\mathcal{A}) \mid \mathrm{lab}(r) = u)$ (see Section 2.3). For a language K let $\mathsf{Run}(\mathcal{A}, K) = \bigcup_{u \in K} \mathsf{Run}(\mathcal{A}, u)$. We define a linear order $\sqsubseteq$ on $\mathsf{Run}(\mathcal{A})$ as follows, where we fix an arbitrary linear order on the transitions of $\mathcal{A}$ (so that runs of $\mathcal{A}$ can be ordered lexicographically): For $r, r' \in \mathsf{Run}(\mathcal{A})$ let $r \sqsubseteq r'$ if and only if $\mathrm{lab}(r) <_{\mathsf{lex}} \mathrm{lab}(r')$ or $\mathrm{lab}(r) = \mathrm{lab}(r')$ and $r \leq_{\mathsf{lex}} r'$. Then $(\mathsf{Run}(\mathcal{A}); \sqsubseteq)$ is an automatic linear order that is obtained from $(\{0,1\}^*1\#D\#E; \leq_{\mathsf{lex}})$ by replacing every word $u\#v\#w$ by a finite linear order on the set of runs accepting $u\#v\#w$. We show in four steps that it is isomorphic to $\mathsf{aut}(P)$:

Step 1: Let $u\#v\#w \in \{0,1\}^*1\#D\#E$. Then the set $\mathsf{Run}(\mathcal{A}, u\#v\#w)$ of runs $r \in \mathsf{Run}(\mathcal{A})$ accepting $u\#v\#w$ is a finite interval in $(\mathsf{Run}(\mathcal{A}); \sqsubseteq)$ whose size is given by (17).

Step 2: Let $u\#v \in \{0,1\}^*1\#D$. We analyze the restriction of the linear order $(\mathsf{Run}(\mathcal{A}); \sqsubseteq)$ to the set $\mathsf{Run}(\mathcal{A}, u\#v\#E)$. Since $u\#v\#E$ is an interval in $(\{0,1\}^*1\#D\#E; \leq_{\mathsf{lex}})$, so is $\mathsf{Run}(\mathcal{A}, u\#v\#E)$ in $(\mathsf{Run}(\mathcal{A}); \sqsubseteq)$. It is obtained from $(u\#v\#E; \leq_{\mathsf{lex}})$ by replacing $u\#v\#w$ by a linear order of size $|\mathsf{Run}(\mathcal{A}, u\#v\#w)|$ (which is given by (17)). The size of this linear order depends on $\mathrm{col}_E(w)$ and $\mathrm{col}_D(v)$ (but $\mathrm{col}_D(v) \in \mathbb{N}$ is constant since we fixed v). Hence, by Lemma 53, any size that appears at all appears densely, i.e., $(\mathsf{Run}(\mathcal{A}, u\#v\#E); \sqsubseteq)$ is the shuffle sum of the class of finite linear orders

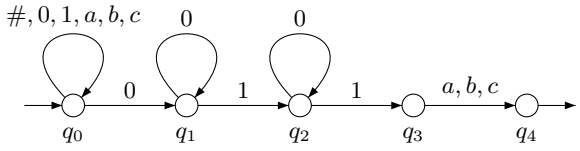


Fig. 11. The automaton $\mathcal{A}_{z_\ell}$

$\mathcal{L}_{u\#v} := \{\mathbf{n} \mid \exists w \in E : n = |\mathsf{Run}(\mathcal{A}, u\#v\#w)\}$. To determine this class precisely, we distinguish the cases $\mathrm{col}_D(v) = 0$ and $\mathrm{col}_D(v) > 0$.

If $\mathrm{col}_D(v) = 0$, then $\mathcal{L}_{u\#v}$ consists of the linear orders $L[p_1(\mathsf{num}(u), \overline{n}) + n_{\ell+1}, p_2(\mathsf{num}(u), \overline{n}) + n_{\ell+1}]$ (recall that $L[x, y]$ denotes the finite linear order with $C(x, y)$ many elements) for $\overline{n} \in \mathbb{N}_+^\ell$ and $n_{\ell+1} \in \mathbb{N}_+$ and the linear orders $L[m, n]$ for $m \neq n$. If $u \in P$, then $p_1(\mathsf{num}(u), \overline{n}) + n_{\ell+1} \neq p_2(\mathsf{num}(u), \overline{n}) + n_{\ell+1}$ for all values of $\overline{n}$ and $n_{\ell+1}$ and therefore $\mathcal{L}_{u\#v} = \mathcal{L}'_\omega$, where the set $\mathcal{L}'_\omega$ was defined in (15). If $u \notin P$, let $\kappa \in \mathbb{N}_+$ be minimal with $p_1(\mathsf{num}(u), \overline{n}) + n_{\ell+1} = \kappa = p_2(\mathsf{num}(u), \overline{n}) + n_{\ell+1})$ for some values of $\overline{n}$ and $n_{\ell+1}$. Then $\mathcal{L}_{u\#v} = \mathcal{L}'_\kappa$ and $\kappa \geq 2$ since it is the sum of two positive integers $p_1(\mathsf{num}(u), \overline{n})$ and $n_{\ell+1}$.

On the other hand, if $\mathrm{col}_D(v) > 0$, then the set $\mathcal{L}_{u\#v}$ consists of the finite linear orders $L[\mathrm{col}_D(v) + n, \mathrm{col}_D(v) + n]$ for $n \in \mathbb{N}_+$ and $L[m, n]$ for $m \neq n$. Hence $\mathcal{L}_{u\#v} = \mathcal{L}'_{\mathrm{col}_D(v)+1}$.

In summary,

$$(\mathsf{Run}(\mathcal{A}, u\#v\#E); \sqsubseteq) \cong \begin{cases} L'_\omega & \text{if } \mathrm{col}_D(v) = 0 \text{ and } u \in P \\ L'_\kappa & \text{for some } 2 \leq \kappa < \omega \text{ if } \mathrm{col}_D(v) = 0 \text{ and } u \notin P \\ L'_{c(v)+1} & \text{if } \mathrm{col}_D(v) > 0 \end{cases} \tag{18}$$

Step 3: Next, let $u \in \{0, 1\}^*1$. We analyze the restriction of $(\mathsf{Run}(\mathcal{A}); \sqsubseteq)$ to the set $\mathsf{Run}(\mathcal{A}, u\#D\#E)$. Since $u\#D\#E$ is an interval in $(\{0, 1\}^*1\#D\#E; \leq_{\mathsf{lex}})$, so is $\mathsf{Run}(\mathcal{A}, u\#D\#E)$ in $(\mathsf{Run}(\mathcal{A}); \sqsubseteq)$. It is obtained from $(u\#D; \leq_{\mathsf{lex}})$ by replacing $u\#v$ by the linear order $(\mathsf{Run}(\mathcal{A}, u\#v\#E); \sqsubseteq)$ whose type is given by (18). Since u is fixed, this type depends on $\mathrm{col}_D(v)$ only such that, by Lemma 53, any type that appears at all appears densely, i.e., $(\mathsf{Run}(\mathcal{A}, u\#D\#E); \sqsubseteq)$ is the shuffle sum of the class $\{\mathcal{L}'_\kappa \mid 2 \leq \kappa < \omega\}$ if $u \notin P$ and $\{\mathcal{L}'_\kappa \mid 2 \leq \kappa \leq \omega\}$ otherwise. Hence,

$$(\mathsf{Run}(\mathcal{A}, u\#D\#E); \sqsubseteq) \cong \begin{cases} M_1 & \text{if } u \in P \\ M_0 & \text{otherwise.} \end{cases} \tag{19}$$

Step 4: Finally, $(\mathsf{Run}(\mathcal{A}); \sqsubseteq)$ is obtained from $(\{0, 1\}^*1; \leq_{\mathsf{lex}})$ by replacing u by the finite linear order $(\mathsf{Run}(\mathcal{A}, u\#D\#E); \sqsubseteq)$ whose type is given by (19). Hence, indeed $(\mathsf{Run}(\mathcal{A}); \sqsubseteq) \cong \mathsf{aut}(P)$.

Thus, we proved the following statement:

Proposition 55. *From an index of a Π^0_1-set $P \subseteq \{0, 1\}^*1$, one can compute an automatic presentation of the linear order* $\mathsf{aut}(P)$.

From Propositions 45, 52, 55, and the fact that the isomorphism problem for computable linear orders is Σ^1_1-complete, it follows that the isomorphism problem for automatic linear orders is Σ^1_1-complete as well. In fact, we can sharpen this result even to automatic linear orders of Hausdorff rank 1, which we define next.

For a linear order $L = (A; \leq)$ we define the equivalence relation $\equiv_L$ on A by: $a \equiv_L b$ if and only if the interval (a, b) is finite. Every equivalence class of $\equiv_L$ is an interval of L of order type $\mathbf{n}$ ($n \in \mathbb{N}$), ω, ω^*, or ζ. We define the *finite condensation* of L as the linear order $C(L)$, whose domain

is the set of equivalence classes of $\equiv_L$ and $[a] \leq [b]$ if and only if $a \leq b$. In other words, $C(L)$ results from L by identifying two elements a, b of L if the interval (a, b) is finite. In [24] it was shown that for every automatic linear order L there exists a natural number n such that $C^n(L) = C^{n+1}(L)$; the least such n is called the *Hausdorff rank* of L. A linear order L has Hausdorff rank 1, if after identifying all $a, b \in L$ such that the interval (a, b) is finite, one obtains a dense order or the singleton linear order. The result of [24] mentioned above suggests that the isomorphism problem might be simpler for linear orders of low Hausdorff rank. But this is not the case:

Theorem 56. *There is a linear order L of Hausdorff rank 1 such that the set of automatic presentations of L is Σ^1_1-complete. In particular, the isomorphism problem for automatic linear orders is Σ^1_1-complete.*

Proof. There is a computable linear order, for which the set of all computable copies is Σ^1_1-complete. A concrete example is the Harrison ordering $\omega_1^{\mathrm{CK}}(1 + \eta)$ (here ω_1^{CK} is the Church-Kleene ordinal and η is the order type of the rationals). Let h be an index of $\omega_1^{\mathrm{CK}}(1 + \eta)$ and set $L = \mathsf{aut}(P(h))$. Note that the linear orders L'_κ from (16) have Hausdorff rank 1. Hence M_0 and M_1 are dense sums of linear orders of Hausdorff rank 1, i.e., their Hausdorff rank is 1 as well. Hence, L is a dense sum of orders of Hausdorff rank 1 as well, implying that the Hausdorff rank of L is 1 too.

We reduce the Σ^1_1-complete set of indices of $\omega_1^{\mathrm{CK}}(1 + \eta)$ [13] to the set of automatic presentations of L: Let e be an index of some linear order K. By Proposition 45, we can compute an index for the set $P(e)$. From this index, we can compute an automatic presentation of $\mathsf{aut}(P(e))$ by Proposition 55. Then $K \cong L$ if and only if $(\{0,1\}^*1; \leq, P(e)) \cong (\{0,1\}^*1; \leq, P(h))$ by Proposition 45 if and only if $\mathsf{aut}(P(e)) \cong \mathsf{aut}(P(h)) = L$ by Proposition 52. □

Theorem 57. *The set of automatic presentations of linear orders that are elementary equivalent to L'_ω is Π^0_1-complete. In particular, the elementary equivalence problem for automatic linear orders is Π^0_1-complete.*

Proof. Let $P \subseteq \{0,1\}^*1$ be Π^0_1-complete. For $u \in \{0,1\}^*1$, write $\mathsf{aut}(u)$ for the linear order $(\mathsf{Run}(\mathcal{A}, u\#1\#E); \sqsubseteq)$. By (18), this linear order is isomorphic to L'_ω if $u \in P$ and isomorphic to some L'_m for $m < \omega$ if $u \notin P$. Note that L'_m contains some maximal finite interval of size $C(m, m)$ and L'_ω does not. Hence, for every $m < \omega$ there is a first-order sentence that distinguishes L'_m and L'_ω. Thus, $\mathsf{aut}(u) \equiv L'_\omega$ if and only if $u \in P$. □

5.3 Scattered linear orders

Recall that a linear order L is *scattered* if $(\mathbb{Q}; \leq)$ cannot be embedded into L. Typical examples of scattered linear orders are finite linear orders, ω, ω^*, and ζ. In [24] it was shown to be decidable, whether a given automatic linear order is scattered. In this section, we show that the isomorphism problem for scattered automatic linear orders is simpler than for general linear orders. An important tool in this proof are ordered trees (not to be confused with order trees). An *ordered tree* is a relational structure $T = (V; \leq, R)$, where $(V; \leq)$ is an order tree in the sense of Section 4 and the binary relation R is a disjoint union $\biguplus_{v \in V} \leq_v$, where $\leq_v$ is a linear order on the set of children of v. For a node v, let $E(v)$ be the set of children and write $R(v)$ for the linear order $(E(v); \leq_v)$. An ordered tree $T = (V; \leq, R)$ is *discrete* if, for every $v \in V$, the linear order $R(v)$ is finite or isomorphic to ω, to ω^*, or to ζ.

For $\ell \in \mathbb{N}$, let $\mathcal{O}_\ell$ denote the set of automatic presentations of discrete ordered trees of height at most ℓ and let $\mathcal{O} = \bigcup_{\ell \in \mathbb{N}} \mathcal{O}_\ell$. Note that one can axiomatize in the logic FSO the order type of ω (there are infinitely many elements but for every x the set of all y with $y < x$ is finite). Hence, one can axiomatize the order types ω^* and $\zeta = \omega^* + \omega$ as well. Theorem 4 implies that the class $\mathcal{O}_\ell$ is decidable for every $\ell \in \mathbb{N}$. By Theorem 10, even $\mathcal{O}$ is decidable.

Lemma 58. *The isomorphism problem for the class $\mathcal{O}$ of discrete automatic ordered trees of finite height can be reduced to $\mathsf{FOTh}(\mathbb{N}; +, \times)$.*

Proof. From $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{O}$, we can compute $\ell \in \mathbb{N}$ with $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{O}_\ell$. If $\ell = 0$, then $(\mathcal{P}_1, \mathcal{P}_2) \in \mathcal{O}$. Now suppose $\ell > 0$ and let $T_i = (V_i; \leq_i, R_i)$ be the discrete ordered tree represented by $\mathcal{P}_i$. Let r_i be the root of T_i. For $v \in V_i$, let $\mathcal{P}_i(v)$ be an automatic presentation for the ordered subtree of T_i rooted at v (it can be computed from $\mathcal{P}_i$ and v). Furthermore, let $v+1$ be the right sibling (i.e., the next element in the linear order $R(w)$ where w is the parent node of v) and $v-1$ the left sibling. Inductively, we define $v + (n+1) = (v+n)+1$ and $v - (n+1) = (v-n)-1$ for $n \in \mathbb{N}_+$. Note that $v+n$ and $v-n$ need not be defined, and that we can decide whether $v+n$ is defined since the tree T_i is automatic. To simplify notation in the formula below, let $\mathcal{P}_i(v+n)$ be an automatic presentation for the empty tree in case $v+n$ is not defined. Then we have $(\mathcal{P}_1, \mathcal{P}_2) \in \mathsf{Iso}(\mathcal{O})$ if and only if

$$(\mathcal{P}_1, \mathcal{P}_2) \in \mathsf{Iso}(\mathcal{O}_{\ell-1}) \ \vee \ \exists v_1 \in E(r_1)\, \exists v_2 \in E(r_2)\, \forall n \in \mathbb{Z} : (\mathcal{P}_1(v_1+n), \mathcal{P}_2(v_2+n)) \in \mathsf{Iso}(\mathcal{O}_{\ell-1})$$

By induction, this is a formula from $\Sigma^0_{2\ell}$. □

Theorem 59. *The isomorphism problem for scattered automatic linear orders can be reduced to* $\mathsf{FOTh}(\mathbb{N}; +, \times)$.

Proof. We reduce this isomorphism problem to the isomorphism of automatic discrete ordered trees of finite height. Let $L = (A; \leq)$ be a scattered automatic linear order. Let n be the Hausdorff rank of L; it is finite by [24]. Since L is scattered, we have $C^n(L) \cong \mathbf{1}$. By induction on n, we define a discrete ordered tree T_L as follows: The leaves of T_L are the elements of L. If $n = 0$, then L is a linear order with a single element and T_L is a single node tree. Now assume that $n > 0$ and that the discrete ordered tree $T_{C(L)}$ is already defined. Then, the leaves of $T_{C(L)}$ are the equivalence classes w.r.t. $\equiv_L$. We obtain T_L be attaching to each equivalence class B the elements of B as new children and order them by $\leq$. Recall that the order type of L restricted to an equivalence classes w.r.t. $\equiv_L$ is indeed either finite, ω, ω^*, or ζ. Hence, T_L is a discrete ordered tree. Moreover, since L is automatic, the ordered tree T_L is effectively automatic since the equivalence relation $\equiv_L$ is definable in FSO. Finally, two scattered automatic linear orders L_1 and L_2 are isomorphic if and only if $T_{L_1} \cong T_{L_2}$. □

While the above theorem shows that the isomorphism problem for scattered linear orders is substantially simpler than for arbitrary linear orders, we still do not have any lower bound. We do not even know whether this problem is decidable or not.

5.4 Context-free languages with lexicographic orders

Given two regular languages L_1 and L_2 and a linear order on their alphabet, it is decidable whether $(L_1; \leq_{\mathsf{lex}}) \cong (L_2; \leq_{\mathsf{lex}})$ [36]. For context-free languages, the same problem is undecidable [10] and Ésik asks whether the problem becomes decidable for deterministic context-free languages. In the light of the current paper, it is natural to ask for the exact recursion-theoretic level of undecidability. In this section, we show that it is Σ^1_1-complete for deterministic context-free languages.

We use $\mathcal{L}^{\mathrm{dcf}}_{\mathrm{lex}}$ to denote the class of linear orders isomorphic to some $(L; \leq_{\mathsf{lex}})$ where L is a deterministic context-free language and $\leq_{\mathsf{lex}}$ is a lexicographic order. Each member of the class $\mathcal{L}^{\mathrm{dcf}}_{\mathrm{lex}}$ is represented by a deterministic pushdown automaton P recognizing L. We use $L(P)$ to denote the language recognized by P.

Theorem 60. *There is a linear order L of Hausdorff rank 1 such that the set of deterministic pushdown automata P with $(L(P); \leq_{\mathsf{lex}}) \cong L$ is Σ^1_1-complete. In particular, the isomorphism problem for the class of linear orders $\mathcal{L}^{\mathrm{dcf}}_{\mathrm{lex}}$ is Σ^1_1-complete.*

Proof. It suffices to prove an analogue of Proposition 55. Using the notations from the proof of that proposition, we have to construct a pushdown automaton P such that $(L(P); \leq_{\mathsf{lex}}) \cong (\mathsf{Run}(\mathcal{A}); \sqsubseteq)$. Recall that $(\mathsf{Run}(\mathcal{A}); \sqsubseteq)$ results from $(\{0,1\}^*1\#D\#E; \leq_{\mathsf{lex}})$ by replacing every word w by the finite linear order with $|\mathsf{Run}(\mathcal{A}, w)|$ many elements.

For a word $w = w_1 \cdots w_k$, let $\overleftarrow{w}$ be its *reversal*, i.e., $\overleftarrow{w} = w_k \cdots w_1$. Then consider the language

$$K = \{w\overleftarrow{r} \mid w \in \{0,1\}^*1\#D\#E, r \in \mathsf{Run}(\mathcal{A}, w)\} \subseteq \Sigma^+\Delta^+$$

where $\Sigma = \{0, 1, \#, a, b, c\}$ and Δ is the set of transitions of $\mathcal{A}$. One can easily construct a deterministic pushdown automaton P with $L(P) = K$. On $\Sigma \cup \Delta$ we fix an order with $\delta < \# < 0 < 1 < a < b < c$ for all $\delta \in \Delta$. For any $w \in \{0,1\}^*1\#D\#E$, we have

$$(\mathsf{Run}(\mathcal{A}, w); \sqsubseteq) \cong \mathbf{n} \cong (\{\overleftarrow{r} \mid r \in \mathsf{Run}(\mathcal{A}, w)\}; \leq_{\mathsf{lex}})$$

where n is the number of accepting runs of $\mathcal{A}$ on w. Moreover, since every symbol from Δ is smaller than every symbol from Σ, we have the following for all $w_1, w_2 \in \{0,1\}^*1\#D\#E$ and $r_1 \in \mathsf{Run}(\mathcal{A}, w_1)$, $r_2 \in \mathsf{Run}(\mathcal{A}, w_2)$ with $w_1 \neq w_2$: $w_1\overleftarrow{r_1} \leq_{\mathsf{lex}} w_2\overleftarrow{r_2}$ if and only if $w_1 \leq_{\mathsf{lex}} w_2$. Hence, $(L(P); \leq_{\mathsf{lex}})$ can be obtained from $(\{0,1\}^*1\#D\#E; \leq_{\mathsf{lex}})$ by replacing every word w by the finite linear order with $|\mathsf{Run}(\mathcal{A}, w)|$ many elements. Thus, we have $(K; \leq_{\mathsf{lex}}) \cong (\mathsf{Run}(\mathcal{A}); \sqsubseteq)$. □

6 Conclusion

This paper looks at the isomorphism problem of some typical classes of automatic structures. Such classes include equivalence structures, order trees, and linear orders. We have shown that (i) the isomorphism problem for automatic equivalence structures is Π^0_1-complete and (ii) the isomorphism problem for automatic order trees and linear orders is Σ^1_1-complete. For order trees, we proved better complexity bounds under certain restrictions. For instance, we have shown that the isomorphism problem for automatic well-founded order trees and automatic trees of bounded height is recursively equivalent to first-order arithmetic. For automatic trees of height $n \geq 2$, the isomorphism problem turned out to be Π^0_{2n-3}-complete.

We also showed that the isomorphism problem for scattered linear orders can be reduced to true arithmetic, but any lower bound for this problem is missing.

We conclude with an application of Theorems 37 and 56. The following corollary shows that although automatic structures look simple (especially for automatic trees), there may be no "simple" isomorphism between two automatic copies of the same structure. Recall that the set of hyperarithmetical sets is $\Sigma^1_1 \cap \Pi^1_1$, where Π^1_1 is the set of complements of Σ^1_1 sets. This class is stratified into levels Δ^0_α for each computable ordinal α (the levels $\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$ for $n \in \mathbb{N}$ constitute an initial part of this hierarchy), see [30] for more details. An isomorphism f between two automatic structures with domains L_1 and L_2, respectively, is a Δ^0_α-isomorphism, if the set $\{(x, f(x)) \mid x \in L_1\}$ belongs to Δ^0_α.

Corollary 61. *For every level Δ^0_α of the hyperarithmetical hierarchy, there exist two isomorphic automatic order trees (and two automatic linear orders) without any Δ^0_α-isomorphism.*

This corollary follows by standard arguments from Theorem 43 and 56.

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