# A natural infinite membrane hierarchy

## Henning Fernau





## fernau@uni-trier.de

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## History of the paper

- K. G. Subramanian visited Trier (Grande Région) in January 2013.
- Turning his IWCIA 2012 paper to a journal paper was one of his to-dos.
- This project brought others into play, so that the list of authors is pretty long: Henning Fernau, Rudolf Freund, Markus L. Schmid, ← present here!
   K. G. Subramanian and Petra Wiederhold
- Main result of this project:
  A natural infinite membrane hierarchy

## **Overview on the talk**

- Array grammars with contextual rules
- Membrane mechanisms
- Infinite hierarchies: a riddle on membranes
- Combs: a solution to the riddle
- Discussions

## **Arrays / Pictures**

A *two-dimensional array or a picture* is a function  $\alpha : \mathbb{Z}^2 \to V \cup \{\#\}$  with finite and connected support  $supp(\alpha)$ , given by  $supp(\alpha) = \{v \in \mathbb{Z}^2 | \alpha(v) \neq \#\}$ .  $V^{+2}$ : The set of all two-dimensional arrays over the alphabet V.

### An example:

а							More formally, restricting on the support:
а							$\{((0,0),a),((1,0),a),((2,0),a),$
а							((3,0),a),((4,0),a),((5,0),a),
а							((6,0),a),((1,0),a),((0,2),a), ((6,0),a),((0,1),a),((0,2),a),
а							((0, 0), a), ((0, 1), a), ((0, 2), a),
а	a	а	а	а	а	а	$((0,3),a),((0,4),a),((0,5),a)\}$

Mostly, exact positions are irrelevant  $\sim$  equivalence classes under translation.

### **Contextual Grammars**

- a classical mechanism invented by S. Marcus about 50 years ago;
- the central idea is that of *adjoining* fitting pieces;
  "fitness" can be tested by using *selectors*;
  if fitting, the *context* can be adjoined.
- this has been adapted also to the case of arrays before.

**Contextual Array Rules** have the form  $p = (\alpha, \beta)$ , where

- $\alpha$  is a function defined on  $U_{\alpha} \subset \mathbb{Z}^2$  with values in *V*;
- $\beta$  is a function defined on  $U_{\beta} \subset \mathbb{Z}^2$  with values in V;
- $U_{\alpha} \cap U_{\beta} = \emptyset$ ,  $U_{\alpha}, U_{\beta}$  are finite.

 $(U_{\alpha}, \alpha)$  is called the *selector* and  $(U_{\beta}, \beta)$  the *context* of the production  $(\alpha, \beta)$ ;  $U_{\alpha}$  is called the *selector area*, and  $U_{\beta}$  is the *context area*.  $C_1 \Longrightarrow_p C_2$ : find an occurrence of *p*'s selector as a sub-array in  $C_1$ ; if then the context area is all blanks, replace it by the context to obtain  $C_2$ .

### A small example

Consider the rule:  $p_1 := \begin{bmatrix} a & a \\ a & a \end{bmatrix}$ Here, the selector is shown by box-framing some symbols.

	a		
Consider the array	a	a c	а
Consider the array	a		а
	a	а	а

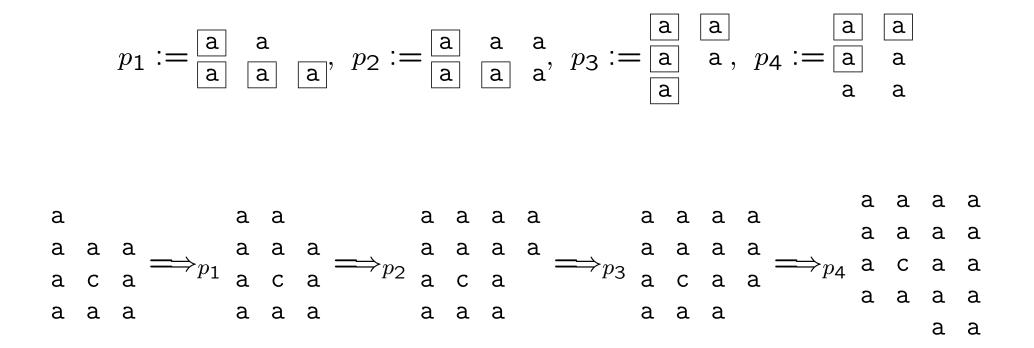
The selector occurs twice, but only in the green situation, the context can be inserted.

**Contextual Array Grammars** are constructs G = (V, P, A) where

- V is an alphabet,
- A is a finite set of two-dimensional arrays in  $V^{+2}$ , called *axioms*,
- and *P* is a finite set of contextual array rules.
- $\mathcal{C}_1 \Longrightarrow_G \mathcal{C}_2$  means  $\exists p \in P : \mathcal{C}_1 \Longrightarrow_p \mathcal{C}_2$ .
- $\mathcal{A} \Longrightarrow_{G}^{t} \mathcal{B}$  if and only if  $\mathcal{A} \Longrightarrow_{G}^{*} \mathcal{B}$  and there is no  $\mathcal{C} \in V^{+2}$  with  $\mathcal{B} \Longrightarrow_{G} \mathcal{C}$ .

• 
$$L_t(G) = \{ \mathcal{B} \in V^{+2} \mid \mathcal{A} \Longrightarrow^t_G \mathcal{B} \text{ for some } \mathcal{A} \in A \}.$$

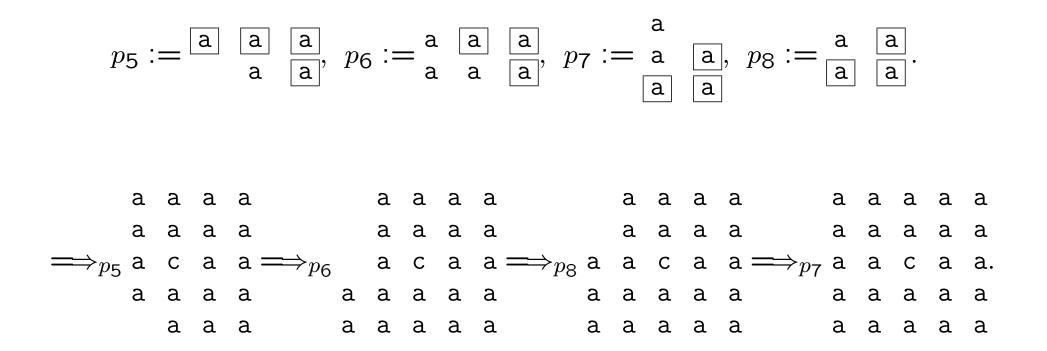
#### A small example



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... continues ... and stops (*t*-mode)



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Contextual Array Grammars are (in a sense) very weak.

- The simple one-dimensional array language consisting of all "lines" of thickness one over the alphabet a cannot be generated.
- Instead, we need a *stopper* at the end.
- Even with stopper(s), L-shaped arrays (thickness one, equal "arm lengths") cannot be described.

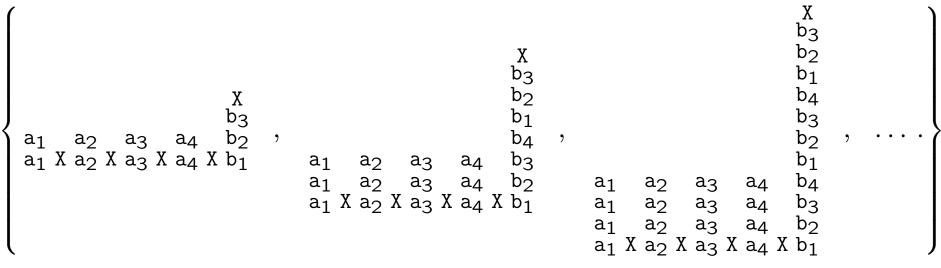
 $\rightsquigarrow$  We need some sort of control mechanism. The IWCIA paper leads us to P systems. This can be also seen as a kind of tree control.

## Membrane computing on array structures

- The membrane structure can be described by labeled brackets.
- Given by a construct  $\Pi = (V, \#, \mu, A_1, \dots, A_m, P_1, \dots, P_m, i_o),$
- $A_i$ : axiom set of membrane *i*.
- $P_i$ : rule set of membrane *i*; each rule has a direction:
- targets *here*, *out*,  $in_j$ ,  $1 \le j \le m$ , and *in*.
- $i_o$  is the *output membrane* for the results of the computation.

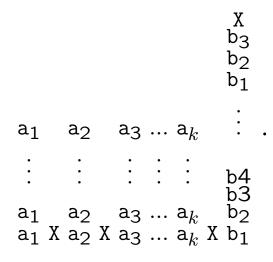
#### Combs

The language of combs with 4 teeth (where the  $i^{\text{th}}$  tooth is defined over the alphabet  $\{a_i\}, 1 \leq i \leq 4$ ) denoted by  $L_{\text{comb},4}$ , is the following set



#### Where is the stopper?

 $L_{\text{comb},k}$  contains combs like in  $L_{\text{comb},4}$ , but with k instead of 4 teeth.



A P system with k + 1 membranes that generates the language  $L_{comb,k}$ 

 $\Pi_{\text{comb},k} = (\{a_1, \dots, a_k, b_1, \dots, b_k, X\}, \#, \mu, A_1, \dots, A_{k+1}, P_1, \dots, P_{k+1}, 1),$ where  $\mu = [1 [2]_2 [3]_3 \dots [k+1]_{k+1} ]_1$ . The sets of axioms are defined by

 $A_1 = \{ a_1 X a_2 X a_3 \dots a_k X \} \text{ and } A_2 = A_3 = \dots = A_{k+1} = \emptyset,$ and the rules are defined by

$$\begin{split} P_1 &:= \{p_1, p'_1, p_i, p'_k \mid 2 \leq i \leq k\} \text{ and } P_i := \{q_i\}, 2 \leq i \leq k+1, \text{ where} \\ p_1 &:= \begin{pmatrix} b_1 \\ b_k \end{pmatrix}, \text{ in}_2 \end{pmatrix}, \quad p'_1 := \begin{pmatrix} \mathbb{X} \\ b_1, \text{ in}_2 \end{pmatrix} \text{ THE START}, \\ p_i &:= \begin{pmatrix} b_i \\ b_{i-1} \end{pmatrix}, \text{ in}_{i+1} \end{pmatrix}, \text{ for every } i, 2 \leq i \leq k, \\ p'_k &:= \begin{pmatrix} \mathbb{X} \\ b_{k-1} \end{pmatrix}, \text{ in}_{k+1} \end{pmatrix} \text{ THE END}, \\ q_i &:= \begin{pmatrix} a_{i-1} \\ a_{i-1} \end{pmatrix}, \text{ out} \end{pmatrix}, \text{ for every } i, 2 \leq i \leq k+1. \end{split}$$

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## **Riddles for membrane computing**

- Question (by Gh. Păun): Find an infinite non-universal hierarchy of languages induced by the number of membranes of the involved P systems.
- Ibarra's answer from 2003: The Number of Membranes Matters. His construction is based on results on multihead finite automata (exhibiting P system simulations via counters).
- A kind of earlier answer from 2001 by R. Freund.
- Here, we provide an alternative answer, based on well-established grammar models, using some interchange argument.

## A natural infinite membrane hierarchy

 $AP_k(cont)$ : The array languages that can be obtained with k membranes with contextual rules.

Lemma: For any  $k \ge 1$ ,  $L_{\text{comb},k} \in AP_{k+1}(cont)$ .

Main Theorem: For any  $k \ge 1$ ,  $L_{\text{comb},k} \notin AP_{\lfloor \frac{k}{3} \rfloor}(cont)$ .

**Corollary**: The hierarchy of array languages

 $\mathsf{AP}_1(\mathit{cont}) \subseteq \mathsf{AP}_2(\mathit{cont}) \subseteq \mathsf{AP}_3(\mathit{cont}) \subseteq \cdots$ 

is infinite and, for every  $k \ge 1$ ,  $AP_k(cont) \subsetneq AP_{3k}(cont)$ .

Lemma 1: W.I.o.g., there is one axiom of one membrane that is

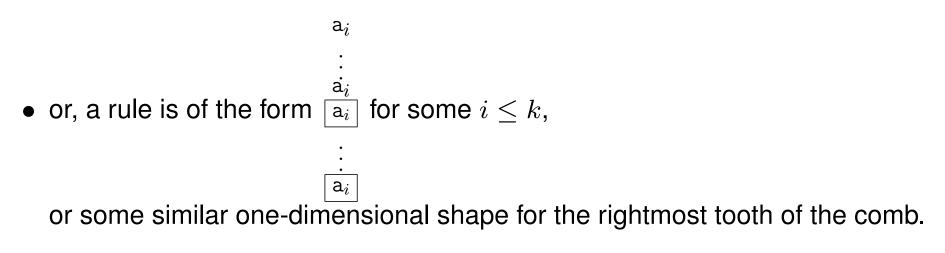
 $a_1 X a_2 X \dots a_k$ 

<u>Idea</u>: If this was not the case, then there is no way of synchronizing thre number of teeth and the length of the last tooth. Of course, the back of the comb could be generated step by step, but this could be shortcut by the suggested axiom.

If there are more and other axioms, this does not change the following arguments.

**Lemma 2**: Let  $\Pi$  be some P system with contextual array rules that generates  $L_{\text{comb},k}$ . Then, each rule of  $\Pi$  can be classified into two groups:

• Either, a rule is X-sensing, meaning that it contains X in its selector;



Idea: A non-X-sensing rule cannot synchronize between different teeth.

Let  $\Pi$  be a P system and let p be a rule of  $\Pi$ . The *type* of rule p is specified by the pair of membranes  $(m_1, m_2)$ , where  $m_1$  is the membrane p belongs to and  $m_2$  is the membrane where the processing will continue (as indicated by the target command  $in_i$ , out or here).

Lemma 3: Any P system  $\Pi$  with m membranes has at most 3m - 2 different rule types. If  $\Pi$  has no rules labelled *here*, then it has rules of at most 2m - 2 different types.

<u>Proof</u>: It is well-known that any undirected tree on m nodes has m - 1 edges. Clearly, the membrane structure of  $\Pi$  corresponds to such a tree. Each edge of this tree may be used in two directions (by the *out* or  $in_j$  labels); this gives (at most) 2m - 2 different types, and each rule labelled *out* or  $in_j$  has one of these types. Rules labelled *here* correspond to loops at some nodes. Hence, altogether  $\Pi$  cannot have more than 3m - 2 different rule types.

## Lemma 4: No P system $\Pi$ with *m* membranes can produce $L_{\text{comb},3m}$ .

<u>Proof</u>: Assume the contrary, so there exists some P system  $\Pi$  with m membranes that can produce  $L_{\text{comb},3m}$ . We can assume that  $\Pi$  is in the normal form described in Lemmas 1 and 2. Let x be the greatest diameter of any X-sensing rule in  $\Pi$ , and y be the greatest diameter of any axiom in  $\Pi$ . Let h > x + y and consider the array  $A_h \in L_{\text{comb},3m}$  whose first tooth contains exactly h occurrences of  $a_1$ .

As  $AL(\Pi) = L_{comb,3m}$  by our assumption, there is some halting computation that outputs  $A_h$ .

Let us discuss the derivation of  $A_h$  in this computation in the following. By the choice of h, for the  $i^{\text{th}}$  tooth, there must be a rule  $p_i$  of vertical one-dimensional shape that extends the  $i^{\text{th}}$  tooth in that derivation. By Lemma 4,  $\Pi$  contains at most 3m-2 rule types. By the pigeon-hole principle, among the rules  $p_1, \ldots, p_{3m-1}$ , there must be two of the same type, say,  $p_I$  and  $p_J$ .

Reconsider now the mentioned computation. W.I.o.g., assume that  $p_I$  was applied later than  $p_J$  was ever applied. However, as also  $p_J$  has been applied (before that time), it is also applicable at that point of time, so we could replace one application of  $p_I$  by one of  $p_J$  to get a new terminating computation that will output an array that is not in  $L_{\text{comb},3m}$ , as the  $I^{\text{th}}$  tooth is now shorter than h, while the  $J^{\text{th}}$  tooth will be longer than h.

## Improving the gap in the Main Theorem

**Remark**: No P system  $\Pi$  with *m* membranes can produce  $L_{\text{comb},2m}$ , by making the following Claim:

In the discussion of the non-X sensing rules working on tooth 1 through k in the proof of Lemma 4, none of these rules has the target indication *here*.

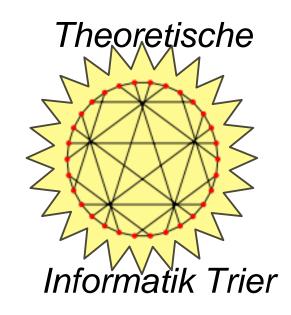
Open Problem: Close the gap perfectly! Prove that  $AP_1(cont) \subsetneq AP_2(cont) \subsetneq AP_3(cont) \subsetneq AP_4(cont) \cdots$ Notice: Ibarra's hierarchy is not dense, either!

We could only show: Lemma 5:  $AP_1(cont) \subsetneq AP_2(cont) \subsetneq AP_3(cont)$ . Separating examples are L- or T-shaped arrays, resp.

#### **Discussions**

- The choice of the regulation mechanism was kind of arbitrary.
- Similar results holds for graph-controlled contextual array grammars of various types.
- More generally, restricting the number of vertices in a graph-control might be an interesting measure of descriptional complexity, related but not identical to the number of productions.

# Thanks for your attention !



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