Automatic structures
Lecture 1: Motivation, definitions, and basic properties

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Motivation

Computable structures
Rational graphs
Automatic structures

Automatic relations

∼ and regular languages
Closure properties of ∼
Computable structures I

Definition
A graph \((V; E)\) is computable if \(V \subseteq \mathbb{N}\) and \(E \subseteq V \times V \subseteq \mathbb{N}^2\) are decidable, i.e., a computable graph is given by a pair of Turing machines \((T_V, T_E)\) that decide \(V\) and \(E\), resp.

Basic problems with this class

- **first-order theory undecidable**: there exists a computable graph whose first-order theory is \(\Delta^0_\omega\)-complete.
- **natural problems are highly undecidable**:
  - the set of pairs \((T_V, T_E)\) representing some graph \(G\) with an infinite clique (with a Hamiltonian path, resp) is \(\Sigma^1_1\)-complete.
  - there exists a graph \(G\) such that the set of presentations of graphs isomorphic to \(G\) is \(\Sigma^1_1\)-complete.
The arithmetical and the analytical hierarchy – officially

- A set $R \subseteq \mathbb{N}$ is in $\Sigma^0_n$ if there exists a polynomial $p \in \mathbb{N}[x, y_1, \ldots, y_n]$ such that
  $$x \in R \iff \exists y_1 \forall y_2 \ldots \exists \forall y_n : p(x, \bar{y}) = 0.$$  
  $$\Pi^0_n = \{ \mathbb{N} \setminus R : R \in \Sigma^0_n \} \neq 2^\mathbb{N} \setminus \Sigma^0_n$$
- A set $R \subseteq \mathbb{N}$ is in $\Delta^0_\omega$ if there exists a computable function $f : \mathbb{N} \to \bigcup_{n \geq 0} \mathbb{N}[x, y_1, \ldots, y_n]$ such that
  $$x \in R \iff \exists y_1 \forall y_2 \ldots \exists \forall y_n : f(x)(x, \bar{y}) = 0.$$  
- A set $R \subseteq \mathbb{N}$ is in $\Sigma^1_1$ if there exists an oracle Turing machine $M$ such that
  $$x \in R \iff \exists X \subseteq \mathbb{N} \forall y \exists z : M^X \text{ accepts } (x, y, z).$$

Relations between these classes

$$\text{REC} \subset \text{RE} = \Sigma^0_1 \subset \Sigma^0_2 \subset \Sigma^0_3 \subset \ldots \Delta^0_\omega \subset \Sigma^1_1 \subset 2^\mathbb{N}$$
The arithmetical and the analytical hierarchy – unofficially

universe $\mathcal{U}$: all finitary objects (e.g. natural numbers, words, automata, finite sets . . .)

relations: all decidable relations on $\mathcal{U}$

$\Sigma^n_0$: all relations defined by formulas of form
$\exists \bar{x}_1 \forall \bar{x}_2 \ldots \exists / \forall \bar{x}_n : R(\bar{x}, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

$\Pi^n_0$: all relations defined by formulas of form
$\forall \bar{x}_1 \exists \bar{x}_2 \ldots \forall / \exists \bar{x}_n : R(\bar{x}, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

$\Delta^n_\omega$: all relations $\{\bar{x} \in \mathcal{U} \mid \mathcal{U} \models f(\bar{x})(\bar{x})\}$ with $f : \mathbb{N}^k \rightarrow \text{FO}[\mathcal{U}]$ computable

$\Sigma^1_1$: all relations defined by formulas of form $\exists X_1, \ldots, X_m : \varphi$ with $\varphi$ first-order, $X_i$ relation variable
Computable structures I

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Computable structures II

Possible solution
restrict class by, e.g., restricting class of admissible presentations – how far?

polynomial time is too powerful
for any computable graph $G$, there exists an isomorphic one $G' = (V'; E')$ such that $V'$ and $E'$ are both in $P$ (and a presentation of $G'$ can be computed from one of $G$).

asynchronous multitape automata are too powerful
see below
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Multitape automata

Some properties

- accept relations on $\Gamma^*$, emptiness decidable
- effective closure under union, projection, cylindrification
- not closed under complementation, intersection; universality undecidable
Rational graphs

A graph \((V; E)\) is **rational** if \(V \subseteq \Sigma^*\) is regular and \(E \subseteq V \times V \subseteq \Sigma^* \times \Sigma^*\) is accepted by some multitape automaton.

clear
rational graphs form a (proper) subclass of all computable graphs (up to isomorphism).

Example subword order

\(V = \{a, b\}^*\) all words – clearly regular
\(E = \{(u, v) \mid u \text{ is subword of } v\}\), e.g.,
\((abba, abbaa), (abba, ababa) \in E\) – accepted by 2-tape automaton with one state

\(K'06\): \(\Sigma_3\)-theory of \((V; E)\) is undecidable.
\(\Rightarrow\) restriction of class of computable structures to rational ones does not suffice.
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Synchronous multitape automata

A relation accepted by $M$: $R(M)$

$R \subseteq (\Gamma^*)^k$ automatic if it is accepted by some synchronous $k$-head automaton

Some properties of automatic relations

- emptiness and universality decidable
- effective closure under union, projection, cylindrification, complementation, intersection
Automatic structures

Definition (Khoussainov & Nerode '95)

A relational structure \((V, (R_i)_{1 \leq i \leq n})\) is

1. regular, if \(V \subseteq \Gamma^*\) and \(R_i \subseteq V^k \subseteq (\Gamma^*)^k\) can be accepted by synchronous \(k\)-tape automata \(M\) and \(M_i\), resp. (For algorithmic purposes, a regular structure \(\mathcal{A}(P)\) is given by a presentation \(P = (M, (M_i)_{1 \leq i \leq n})\))

2. automatic, if it is isomorphic to some regular structure.

Examples of automatic structures

- all finite structures
- complete binary tree, length-lexicographic order \(\leq_{\text{llex}}\)
- Presburger arithmetic \((\mathbb{N}, +)\) (Skolem arithmetic \((\mathbb{N}, \cdot)\) is not automatic)
- \((\mathbb{Q}, \leq)\) (K '03: even automatic-homogeneous)
Automatic structures

Definition (Khoussainov & Nerode '95)

A relational structure $(V, (R_i)_{1 \leq i \leq n})$ is

1. **regular**, if $V \subseteq \Gamma^*$ and $R_i \subseteq V^k \subseteq (\Gamma^*)^k$ can be accepted by synchronous $k$-tape automata $M$ and $M_i$, resp. (For algorithmic purposes, a regular structure $A(P)$ is given by a presentation $P = (M, (M_i)_{1 \leq i \leq n})$)

2. **automatic**, if it is isomorphic to some regular structure.

Examples of automatic structures

- rewrite graph $(\Sigma^*, \rightarrow)$ of semi-Thue system
- configuration graph of a Turing machine
- configuration graph with reachability $(Q\Gamma^*, \rightarrow, \rightarrow^*)$ of a pushdown automaton
Examples

- Cayley-graphs of automatic monoids, in particular of
  - rational monoids (Sakarovitch '87)
  - virtually free f.g., virtually free Abelian f.g., and of hyperbolic groups (Epstein et al. '92)
  - singular Artin monoids of finite type (Corran, Hoffmann, K & Thomas '06)
  - graph products of such monoids (Fohry & K '05)
- ordinal $\alpha$ automatic iff $\alpha < \omega^\omega$ (Delhommé, Goranko & Knapik '03)
- $\mathcal{B} = \text{Boolean algebra of (co-)finite subsets of } \mathbb{N}$
  infinite Boolean algebra automatic iff $\mathcal{B}^n$ for some $n \in \mathbb{N}$ (Khoussainov, Nies, Rubin, Stephan '04)
- field automatic iff finite (Khoussainov, Nies, Rubin, Stephan '04)
- f.g. group automatic iff virtually Abelian (Oliver & Thomas '05)
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\sim\text{ and regular languages}
Closure properties of \sim
Finite automata

A finite automaton over the alphabet $\Sigma$ is a tuple $M = (Q, I, T, F)$ such that

- $Q$ is a finite set of “states”,
- $I \subseteq Q$ is the set of “initial states”,
- $T \subseteq Q \times \Sigma \times Q$ is the set of “transitions”, and
- $F \subseteq Q$ is the set of “accepting” or “final states”.

A run of $M$ is a nonempty word

$$r = (p_0, a_1, p_1)(p_1, a_2, p_2) \ldots (p_{n-1}, a_n, p_n) \in T^+,$$

$p_0$ is its initial state, $p_n$ its final one, and $w = a_1a_2 \ldots a_n \in \Sigma^+$ its label. It is accepting if $p_0 \in I$ and $p_n \in F$.

The language $L(M)$ of $M$ is the set of labels of accepting runs.

A language $L \subseteq \Sigma^+$ is regular if it is the language of some finite automaton.
From tuples of words to words

For a tuple of words \((w_1, w_2, \ldots, w_n)\) over \(\Sigma\) with \(w_i = a_{i1}^i a_{i2}^i \ldots a_{ik_i}^i\), let the convolution be defined by

\[
\bigotimes(w_1, \ldots, w_n) = \begin{pmatrix} b_1^1 \\ b_2^1 \\ \vdots \\ b_n^1 \\ b_1^2 \\ b_2^2 \\ \vdots \\ b_n^2 \\ \vdots \\ b_1^k \\ b_2^k \\ \vdots \\ b_n^k \end{pmatrix} \in (\Sigma \cup \{\diamond\})^n)
\]

with \(k = \max(k_1, k_2, \ldots, k_n)\) and

\[
b_{ij}^i = \begin{cases} a_{ij}^i & \text{if } j \leq k_i \\ \diamond & \text{otherwise} \end{cases}
\]
From relations to languages

For a relation $R \subseteq (\Sigma^*)^n$, let the convolution $\otimes R$ be defined by

$$\otimes R = \{\otimes(w_1, \ldots, w_n) \mid (w_1, \ldots, w_n) \in R\} \subseteq ((\Sigma \cup \{\diamond\})^n)^*$$

Fact

A relation $R$ is automatic (i.e., accepted by some synchronous multitape automaton) if and only if its convolution $\otimes R$ is regular.
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Lemma
If $R_1, R_2 \subseteq (\Sigma^+)^n$ are automatic, then $R_1 \cup R_2$ effectively automatic.

Proof
$M_i = (Q_i, I_i, T_i, F_i)$ finite automaton accepting $\otimes R_i$.

w.l.o.g. $Q_1 \cap Q_2 = \emptyset$.
Then $(Q_1 \cup Q_2, I_1 \cup I_2, T_1 \cup T_2, F_1 \cup F_2)$ accepts $(\otimes R_1) \cup (\otimes R_2) = \otimes (R_1 \cup R_2)$.  

\[\square\]
Complementation

Lemma
If \( R \subseteq (\Sigma^+)^n \) is automatic, then its complement \((\Sigma^+)^n \setminus R\) is effectively automatic.

Proof
\( R \) automatic \( \Rightarrow \otimes R \) regular language in \( \Gamma^+ \) with \( \Gamma = (\Sigma \cup \{\diamond\})^n \)
\( \Rightarrow \Gamma^+ \setminus \otimes R \) regular

The convolution of the complement of \( R \) equals
\[
\Gamma^+ \setminus \otimes R \cap \otimes (\Sigma^+)^n
\]
since \((\Sigma^+)^n\) is automatic, this intersection is regular.
Interlude

1. there are automatic binary relations $R$ and $S$ s.t.
   $R \cdot S = \{(uv, u'v') \mid (u, v) \in R, (u', v') \in S\}$ is not automatic

2. $(R \cap S) = (R^{co} \cup S^{co})^{co}$, hence intersection of automatic relations is effectively automatic, but automaton is huge!
Intersection

Lemma
If $R_1, R_2 \subseteq (\Sigma^+)^n$ are automatic, then $R_1 \cap R_2$ is effectively automatic.

Proof
$M_i = (Q_i, I_i, T_i, F_i)$ finite automaton accepting $\otimes R_i$.

\[
\begin{align*}
Q & := Q_1 \times Q_2 \\
I & := I_1 \times I_2 \\
T & := \{((p, p'), \bar{a}, (q, q')) \mid (p, \bar{a}, q) \in T_1, (q, \bar{a}, q') \in T_2\} \\
F & := F_1 \times F_2
\end{align*}
\]

Then $(Q, I, T, F)$ accepts $(\otimes R_1) \cap (\otimes R_2) = \otimes(R_1 \cap R_2)$. \qed
Projection

Lemma
If \( R \subseteq (\Sigma^+)^n \) is automatic, then its projection 
\( \{(w_1, \ldots, w_{n-1}) \mid \exists w_n : (w_1, \ldots, w_n) \in R\} \) is effectively automatic.

Proof (for \( n = 2 \))
\( M = (Q, I, T, F) \) finite automaton for \( \otimes R \).

\[
T' := \{(p, a, q) \in Q \times \Sigma \times Q \mid \exists b \in \Sigma \cup \{\diamond\} : (p, (a, b), q) \in T\}
\]
\[
F' := \{p \in Q \mid (Q, \{p\}, T, F) \text{ accepts some word from } (\{\diamond\} \times \Sigma)^+\} \cup F
\]

Then \( (Q, I, T', F') \) accepts \( \otimes \{u \mid \exists v : (u, v) \in R\} \). \qed
Cylindrification

Lemma
If $R \subseteq (\Sigma^+)^n$ is automatic, then its cylindrification
$\{(w_1, \ldots, w_n, w_{n+1}) \mid (w_1, \ldots, w_n) \in R, w_{n+1} \in \Sigma^+\}$ is effectively automatic.

Proof (for $n = 1$)
$M = (Q, I, T, F)$ finite automaton for $\otimes R$.

new set of states: $Q' = Q \times \{0, 1\} \cup \{\top\}$
for $(p, a, q) \in T$ and $b \in \Sigma$, transitions in $T'$:
$((p, 0), (a, b), (q, 0)), ((p, 0), (a, \diamond), (q, 1))$, and $((p, 1), (a, \diamond)(q, 1))$
furthermore, transitions $((f, 0), (\diamond, b), \top)$ for $f \in F$ and
$(\top, (\diamond, b), \top)$
$F' = F \times \{0, 1\} \cup \{\top\}$.
Then $(Q', I, T', F')$ accepts $\{(u, v) \mid u \in R\}$. 

See you tomorrow!