Automatic structures
Lecture 1: Motivation, definitions, and basic properties

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Computable structures I

Definition

A graph $(V; E)$ is computable if $V \subseteq \mathbb{N}$ and $E \subseteq V \times V \subseteq \mathbb{N}^2$ are decidable, i.e., a computable graph is given by a pair of Turing machines (T_V, T_E) that decide V and E, resp.

Basic problems with this class

- first-order theory undecidable: there exists a computable graph whose first-order theory is Δ^0_ω-complete.
- natural problems are highly undecidable:
  - the set of pairs (T_V, T_E) representing some graph G with an infinite clique (with a Hamiltonian path, resp) is Σ^1_1-complete.
  - there exists a graph G such that the set of presentations of graphs isomorphic to G is Σ^1_1-complete.

The arithmetical and the analytical hierarchy – officially

- A set $R \subseteq \mathbb{N}$ is in Σ^0_n if there exists a polynomial $p \in \mathbb{N}[x, y_1, \ldots, y_n]$ such that

$$x \in R \iff \exists y_1 \forall y_2 \ldots \exists/\forall y_n : p(x, \overline{y}) = 0.$$

$\Pi^0_n = \{\mathbb{N} \setminus R \mid R \in \Sigma^0_n\} \neq 2^{\mathbb{N}} \setminus \Sigma^0_n$

- A set $R \subseteq \mathbb{N}$ is in Δ^0_ω if there exists a computable function $f : \mathbb{N} \to \bigcup_{n \geq 0} \mathbb{N}[x, y_1, \ldots, y_n]$ such that

$$x \in R \iff \exists y_1 \forall y_2 \ldots \exists/\forall y_n : f(x)(x, \overline{y}) = 0.$$

- A set $R \subseteq \mathbb{N}$ is in Σ^1_1 if there exists an oracle Turing machine M such that

$$x \in R \iff \exists X \subseteq \mathbb{N} \forall y \exists z : M^X \text{ accepts } (x, y, z).$$

Relations between these classes

$$\mathrm{REC} \subsetneq \mathrm{RE} = \Sigma^0_1 \subsetneq \Sigma^0_2 \subsetneq \Sigma^0_3 \subsetneq \cdots \Delta^0_\omega \subsetneq \Sigma^1_1 \subsetneq 2^{\mathbb{N}}$$

The arithmetical and the analytical hierarchy – inofficially

universe $\mathcal{U}$: all finitary objects (e.g. natural numbers, words, automata, finite sets ...)

relations: all decidable relations on $\mathcal{U}$

Σ^0_n: all relations defined by formulas of form $\exists \overline{x}_1 \forall \overline{x}_2 \ldots \exists/\forall \overline{x}_n : R(\overline{x}, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$

Π^0_n: all relations defined by formulas of form $\forall \overline{x}_1 \exists \overline{x}_2 \ldots \forall/\exists \overline{x}_n : R(\overline{x}, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$

Δ^0_ω: all relations $\{\overline{x} \in \mathcal{U} \mid \mathcal{U} \models f(\overline{x})(\overline{x})\}$ with $f : \mathbb{N}^k \to \mathrm{FO}[\mathcal{U}]$ computable

Σ^1_1: all relations defined by formulas of form $\exists X_1, \ldots, X_m : \varphi$ with φ first-order, X_i relation variable

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Computable structures II

Possible solution

restrict class by, e.g., restricing class of admissible presentations – how far?

polynomial time is too powerful

for any computable graph G, there exists an isomorphic one $G' = (V'; E')$ such that V' and E' are both in P (and a presentation of G' can be computed from one of G).

asynchronous multitape automata are too powerful

see below

Motivation

Computable structures

Rational graphs

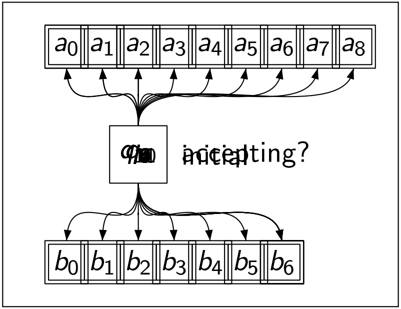
Automatic structures

Automatic relations

$\sim$ and regular languages

Closure properties of $\sim$

Multitape automata



Some properties

- accept relations on Γ^*, emptiness decidable
- effective closure under union, projection, cylindrification
- not closed under complementation, intersection; universality undecidable

Rational graphs

A graph $(V; E)$ is rational if $V \subseteq \Sigma^*$ is regular and $E \subseteq V \times V \subseteq \Sigma^* \times \Sigma^*$ is accepted by some multitape automaton.

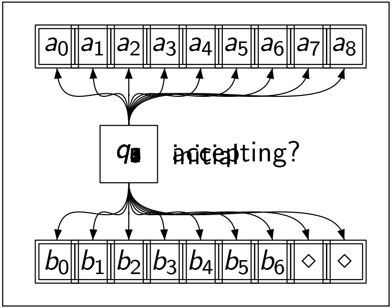
clear

rational graphs form a (proper) subclass of all computable graphs (up to isomorphism).

Example subword order

$V = \{a, b\}^*$ all words – clearly regular
$E = \{(u, v) \mid u \text{ is subword of } v\}$, e.g.,
$(abba, abbaa), (abba, ababa) \in E$ – accepted by 2-tape automaton with one state
K'06: Σ_3-theory of $(V; E)$ is undecidable.
$\Rightarrow$ restriction of class of computable structures to rational ones does not suffice.

Synchronous multitape automata



relation accepted by M: $R(M)$

$R \subseteq (\Gamma^*)^k$ automatic if it is accepted by some synchronous k-head automaton

Some properties of automatic relations

- emptiness and universality decidable
- effective closure under union, projection, cylindrification, complementation, intersection

Automatic structures

Definition (Khoussainov & Nerode '95)

A relational structure $(V, (R_i)_{1 \leq i \leq n})$ is

1. regular, if $V \subseteq \Gamma^*$ and $R_i \subseteq V^k \subseteq (\Gamma^*)^k$ can be accepted by synchronous k-tape automata M and M_i, resp.
   (For algorithmic purposes, a regular structure $\mathcal{A}(P)$ is given by a presentation $P = (M, (M_i)_{1 \leq i \leq n})$)
2. automatic, if it is isomorphic to some regular structure.

Examples of automatic structures

- all finite structures
- complete binary tree, length-lexicographic order $\leq_{\text{llex}}$
- Presburger arithmetic $(\mathbb{N}, +)$ automaton (Skolem arithmetic $(\mathbb{N}, \cdot)$ is not automatic)
- $(\mathbb{Q}, \leq)$ (K '03: even automatic-homogeneous)

Automatic structures

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Examples of automatic structures

- rewrite graph $(\Sigma^*, \rightarrow)$ of semi-Thue system
- configuration graph of a Turing machine
- configuration graph with reachability $(Q\Gamma^*, \rightarrow, \rightarrow^*)$ of a pushdown automaton

Examples

- Cayley-graphs of automatic monoids, in particular of
  - rational monoids (Sakarovitch '87)
  - virtually free f.g., virtually free Abelian f.g., and of hyperbolic groups (Epstein et al. '92)
  - singular Artin monoids of finite type (Corran, Hoffmann, K & Thomas '06)
  - graph products of such monoids (Fohry & K '05)
- ordinal α automatic iff $\alpha < \omega^\omega$ (Delhommé, Goranko & Knapik '03)
- $\mathcal{B}$ = Boolean algebra of (co-)finite subsets of $\mathbb{N}$
  infinite Boolean algebra automatic iff $\mathcal{B}^n$ for some $n \in \mathbb{N}$ (Khoussainov, Nies, Rubin, Stephan '04)
- field automatic iff finite (Khoussainov, Nies, Rubin, Stephan '04)
- f.g. group automatic iff virtually Abelian (Oliver & Thomas '05)

Finite automata

A finite automaton over the alphabet Σ is a tuple $M = (Q, I, T, F)$ such that

- Q is a finite set of "states",
- $I \subseteq Q$ is the set of "initial states",
- $T \subseteq Q \times \Sigma \times Q$ is the set of "transitions", and
- $F \subseteq Q$ is the set of "accepting" or "final states".

A run of M is a nonempty word

$$r = (p_0, a_1, p_1)(p_1, a_2, p_2) \ldots (p_{n-1}, a_n, p_n) \in T^+ ,$$

p_0 is its initial state, p_n its final one, and $w = a_1 a_2 \ldots a_n \in \Sigma^+$ its label. It is accepting if $p_0 \in I$ and $p_n \in F$.
The language $L(M)$ of M is the set of labels of accepting runs.
A language $L \subseteq \Sigma^+$ is regular if it is the language of some finite automaton.

From tuples of words to words

For a tuple of words $(w_1, w_2, \ldots, w_n)$ over Σ with $w_i = a^i_1 a^i_2 \ldots a^i_{k_i}$, let the convolution be defined by

$$\otimes(w_1, \ldots, w_n) = \begin{pmatrix} b^1_1 \\ b^2_1 \\ \vdots \\ b^n_1 \end{pmatrix} \begin{pmatrix} b^1_2 \\ b^2_2 \\ \vdots \\ b^n_2 \end{pmatrix} \cdots \begin{pmatrix} b^1_k \\ b^2_k \\ \vdots \\ b^n_k \end{pmatrix} \in ((\Sigma \cup \{\diamond\})^n)^*$$

with $k = \max(k_1, k_2, \ldots, k_n)$ and

$$b^j_i = \begin{cases} a^j_i & \text{if } j \leq k_i \\ \diamond & \text{otherwise} \end{cases}$$

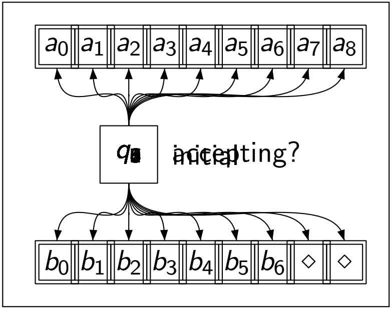
From relations to languages

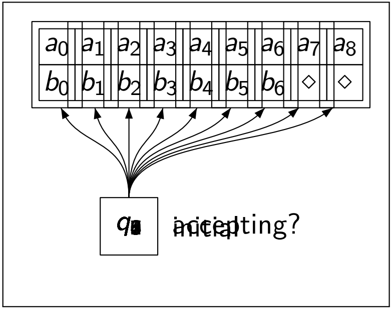
For a relation $R \subseteq (\Sigma^*)^n$, let the convolution $\otimes R$ be defined by

$$\otimes R = \{\otimes(w_1, \ldots, w_n) \mid (w_1, \ldots, w_n) \in R\} \subseteq ((\Sigma \cup \{\diamond\})^n)^*$$

Fact

A relation R is automatic (i.e., accepted by some synchronous multitape automaton) if and only if its convolution $\otimes R$ is regular.





Motivation

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Union

Lemma

If $R_1, R_2 \subseteq (\Sigma^+)^n$ are automatic, then $R_1 \cup R_2$ effectively automatic.

Proof

$M_i = (Q_i, I_i, T_i, F_i)$ finite automaton accepting $\otimes R_i$.
w.l.o.g. $Q_1 \cap Q_2 = \emptyset$.
Then $(Q_1 \cup Q_2, I_1 \cup I_2, T_1 \cup T_2, F_1 \cup F_2)$ accepts
$(\otimes R_1) \cup (\otimes R_2) = \otimes(R_1 \cup R_2)$. $\square$

Complementation

Lemma

If $R \subseteq (\Sigma^+)^n$ is automatic, then its complement $(\Sigma^+)^n \setminus R$ is effectively automatic.

Proof

$$\begin{aligned} R \text{ automatic} &\Rightarrow \otimes R \text{ regular language in } \Gamma^+ \text{ with } \Gamma = (\Sigma \cup \{\diamond\})^n \\ &\Rightarrow \Gamma^+ \setminus \otimes R \text{ regular} \end{aligned}$$

The convolution of the complement of R equals

$$\Gamma^+ \setminus \otimes R \cap \otimes(\Sigma^+)^n$$

since $(\Sigma^+)^n$ is automatic, this intersection is regular. $\square$

Interlude

1. there are automatic binary relations R and S s.t. $R \cdot S = \{(uv, u'v') \mid (u, v) \in R, (u', v') \in S\}$ is not automatic
2. $(R \cap S) = (R^{co} \cup S^{co})^{co}$, hence intersection of automatic relations is effectively automatic, but automaton is huge!

Intersection

Lemma

If $R_1, R_2 \subseteq (\Sigma^+)^n$ are automatic, then $R_1 \cap R_2$ is effectively automatic.

Proof

$M_i = (Q_i, I_i, T_i, F_i)$ finite automaton accepting $\otimes R_i$.

$$
\begin{aligned}
Q &:= Q_1 \times Q_2 \\
I &:= I_1 \times I_2 \\
T &:= \{((p,p'), \bar{a}, (q,q')) \mid (p,\bar{a},q) \in T_1, (q,\bar{a},q') \in T_2\} \\
F &:= F_1 \times F_2
\end{aligned}
$$

Then (Q, I, T, F) accepts $(\otimes R_1) \cap (\otimes R_2) = \otimes(R_1 \cap R_2)$. $\square$

Projection

Lemma

If $R \subseteq (\Sigma^+)^n$ is automatic, then its projection $\{(w_1, \ldots, w_{n-1}) \mid \exists w_n : (w_1, \ldots, w_n) \in R\}$ is effectively automatic.

Proof (for $n = 2$)

$M = (Q, I, T, F)$ finite automaton for $\otimes R$.

$$T' := \{(p, a, q) \in Q \times \Sigma \times Q \mid \exists b \in \Sigma \cup \{\diamond\} : (p, (a, b), q) \in T\}$$

$$F' := \{p \in Q \mid (Q, \{p\}, T, F) \text{ accepts some word from } (\{\diamond\} \times \Sigma)^+\} \cup F$$

Then (Q, I, T', F') accepts $\otimes\{u \mid \exists v : (u, v) \in R\}$. $\square$

Cylindrification

Lemma

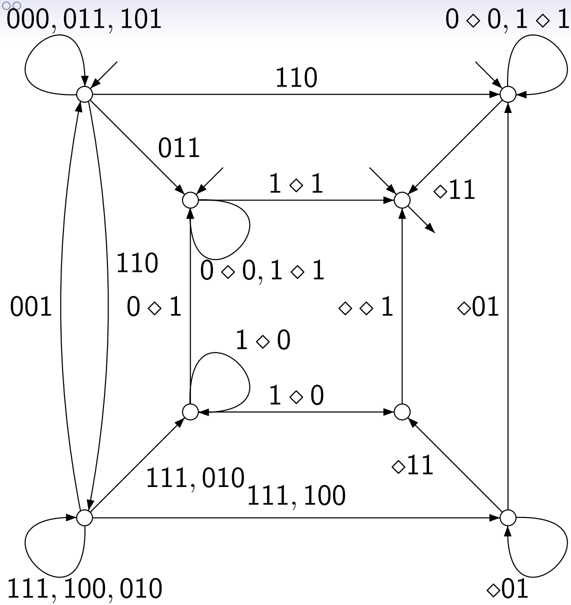
If $R \subseteq (\Sigma^+)^n$ is automatic, then its cylindrification $\{(w_1, \dots, w_n, w_{n+1}) \mid (w_1, \dots, w_n) \in R, w_{n+1} \in \Sigma^+\}$ is effectively automatic.

Proof (for $n = 1$)

$M = (Q, I, T, F)$ finite automaton for $\otimes R$.

new set of states: $Q' = Q \times \{0, 1\} \cup \{\top\}$
for $(p, a, q) \in T$ and $b \in \Sigma$, transitions in T':
$((p, 0), (a, b), (q, 0))$, $((p, 0), (a, \diamond), (q, 1))$, and $((p, 1), (a, \diamond)(q, 1))$
furthermore, transitions $((f, 0), (\diamond, b), \top)$ for $f \in F$ and $(\top, (\diamond, b), \top)$
$F' = F \times \{0, 1\} \cup \{\top\}$.
Then (Q', I, T', F') accepts $\{(u, v) \mid u \in R\}$. $\square$

See you tomorrow!


000, 011, 101
0 ⋄ 0, 1 ⋄ 1
110
011
1 ⋄ 1
⋄11
110
0 ⋄ 0, 1 ⋄ 1
001
0 ⋄ 1
⋄ ⋄ 1
⋄01
1 ⋄ 0
1 ⋄ 0
⋄11
111, 010
111, 100
111, 100, 010
⋄01

return