Automatic structures
Lecture 2: First-order logic and beyond

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First-order logic

Definitions

Let $\mathcal{A} = (V, (R_i)_{1 \le i \le n})$ be a relational structure with $R_i \subseteq V^{k_i}$.

Syntax of FO:

- if $1 \le i \le n$ and $x_1, \ldots, x_{k_i}$ are first-order variables, then $R_i(x_1, \ldots, x_{k_i})$ is a formula of FO
- if x and y are first-order variables, then $x = y$ is a formula
- if α and β are formulas, then so are $\alpha \vee \beta$ and $\neg\alpha$.
- if α is a formula and x a first-order variable, then $\exists x : \alpha$ is a formula.

free variables:

$$\mathrm{var}(R_i(x_1, \ldots, x_{k_i})) = \{x_1, \ldots, x_{k_i}\}$$
$$\mathrm{var}(x = y) = \{x, y\}$$
$$\mathrm{var}(\alpha \vee \beta) = \mathrm{var}(\alpha) \cup \mathrm{var}(\beta)$$
$$\mathrm{var}(\neg\alpha) = \mathrm{var}(\alpha)$$

First-order logic

Definable relations and quotients

Interpretations

The infinity quantifier $\exists^{\infty}$

modulo-quantifiers $\exists^{(p)}$

Second-order quantifiers

FO-definable relations are effectively automatic

Main theorem for FO

From a presentation $P = (M, (M_i)_{1 \leq i \leq n})$ of a regular structure $\mathcal{A}$ and a first-order formula α, one can compute a synchronous multitape-automaton M^{α} such that $R(M^{\alpha}) = \alpha^{\mathcal{A}}$.

Proof

Fix a list of variables $\overline{x}$ containing all variables that appear in α and interprete $\beta^{\mathcal{A}}$ wrt. this list.

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Fix a list of variables $\overline{x}$ containing all variables that appear in α and interprete $\beta^{\mathcal{A}}$ wrt. this list.
By induction on construction of α using closure properties of automatic relations:

$$R_i(y_1, \dots, y_{k_i})^{\mathcal{A}} \text{ is cylindrification of } R_i = R(M_i)$$

Quotients

Let $\mathcal{A} = (V; (R_i)_{1 \leq i \leq n})$ be a relational structure and $\sim \subseteq V^2$ an equivalence relation.
$\sim$ is a congruence if $(u_1, \dots, u_{k_i}) \in R_i$ and $u_j \sim v_j$ imply $(v_1, \dots, v_{k_i}) \in R_i$.

$$R_i/{\sim} := \{([u_1]_\sim, \dots, [u_{k_i}]_\sim) \mid (u_1, \dots, u_{k_i}) \in R_i\}$$

$\mathcal{A}/{\sim} := (V/\sim; (R_i/{\sim})_{1 \leq i \leq n})$ is the quotient of $\mathcal{A}$ wrt. $\sim$

Quotients are automatic

Theorem

From a presentation P of a regular structure $\mathcal{A}$ and an automatic congruence $\sim$, one can compute a presentation of (a regular structure isomorphic to) the quotient $\mathcal{A}/{\sim}$.

Proof

The structure $\mathcal{B} = (\mathcal{A}, \leq_{\text{llex}}, \sim)$ is effectively regular.
The set $\{u \in \mathcal{A} \mid \forall v \in \mathcal{A} : u \sim v \rightarrow u \leq_{\text{llex}} v\}$

- is first-order definable in $\mathcal{B}$ and hence (effectively) regular
- contains precisely one element from every equivalence class of $\sim$.

Hence the restriction of $\mathcal{A}$ to this set is isomorphic to $\mathcal{A}/{\sim}$. □

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First-order interpretations

An *n-dimensional first-order interpretation* consists of

- a structure $\mathcal{A}$,
- a formula ν with n free variables,
- a formula η with $2n$ free variables,
- and formulas ρ_i with $k_i \cdot n$ free variables

such that $\eta^{\mathcal{A}}$ is a congruence of $(\nu^{\mathcal{A}}; (\rho_i^{\mathcal{A}})_{1 \le i \le m})$.

The structure

$$(\nu^{\mathcal{A}}; (\rho_i^{\mathcal{A}})_{1 \le i \le m}) / \eta^{\mathcal{A}}$$

is said to be *interpreted* in $\mathcal{A}$ via $(\nu, \eta, (\rho_i)_{1 \le i \le m})$.

Examples

quotients, direct powers, expansion by definable relations,. . .

First-order interpretations

Corollary

From a presentation P of a regular structure $\mathcal{A}$ and a first-order interpretation I in $\mathcal{A}$, one can compute a presentation of (a regular structure isomorphic to) the structure interpreted in $\mathcal{A}$ via I.

Proof

clear by previous theorems (effective closure under definable expansions and quotients) □

Theorem (Blumensath '99)

A structure is automatic if and only if it can be interpreted in

$$(\{a,b\}^*, \{(u,uv) \mid u,v \in \{a,b\}^*\}, \{(u,v) \mid |u| = |v|\}) .$$

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Infinity quantifier

Let $\mathcal{A} = (V, (R_i)_{1 \leq i \leq n})$ be a relational structure with $R_i \subseteq V^{k_i}$.

Syntax of FO$^\infty$:

- if $1 \leq i \leq n$ and $x_1, \ldots, x_{k_i}$ are first-order variables, then $R_i(x_1, \ldots, x_{k_i})$ is a formula of FO
- if x and y are first-order variables, then $x = y$ is a formula
- if α and β are formulas, then so are $\alpha \vee \beta$ and $\neg\alpha$.
- if α is a formula and x a first-order variable, then $\exists x : \alpha$ and $\exists^\infty x : \alpha$ are formulas.

free variables:

$$\begin{aligned} \mathrm{var}(R_i(x_1, \ldots, x_{k_i})) &= \{x_1, \ldots, x_{k_i}\} \\ \mathrm{var}(x = y) &= \{x, y\} \\ \mathrm{var}(\alpha \vee \beta) &= \mathrm{var}(\alpha) \cup \mathrm{var}(\beta) \\ \mathrm{var}(\neg\alpha) &= \mathrm{var}(\alpha) \end{aligned}$$

FO^∞-definable relations are effectively automatic

Main theorem for FO^∞ (Blumensath '99)

From a presentation $P = (M, (M_i)_{1 \le i \le n})$ of a regular structure $\mathcal{A}$ and a FO^∞-formula α, one can compute a synchronous multitape-automaton M^α such that $R(M^\alpha) = \alpha^{\mathcal{A}}$.

Proof

$(\mathcal{A}, \le_{\text{llex}})$ is effectively automatic and
$\mathcal{A} \models \exists^\infty y : \alpha(y) \iff (\mathcal{A}, \le_{\text{llex}}) \models \forall x \exists y : (x \le_{\text{llex}} y \wedge \alpha(y))$
hence result follows from Main Theorem for FO □

Consequences

The FO^∞-theory of every automatic structure is decidable,
automatic structures are closed under FO^∞-interpretations.

First-order logic

- Definable relations and quotients
- Interpretations

The infinity quantifier $\exists^{\infty}$

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Second-order quantifiers

modulo-quantifiers

Let $\mathcal{A} = (V, (R_i)_{1 \leq i \leq n})$ be a relational structure with $R_i \subseteq V^{k_i}$.

Syntax of FOX:

- formation rules for FO^∞
- if α is a formula, x a first-order variable, and $1 < p$, then $\exists^{(p)} x : \alpha$ is a formula.

free variables:

$$\ldots\ldots$$

$$\mathrm{var}(\exists^{(p)} x : \alpha) = \mathrm{var}(\alpha) \setminus \{x\}$$

semantics of FOX: let $\mathrm{var}(\alpha) \subseteq \{x_1, \ldots, x_n\}$ and $u_1, \ldots, u_n \in V$.
$(\mathcal{A}, (u_1, \ldots, u_{n-1})) \models \exists^{(p)} x : \alpha$ if and only if
$|\{v \in V \mid (\mathcal{A}, (u_1, \ldots, u_{n-1}, v)) \models \alpha\}|$ is finite and divisible by p.

FOX-definable relations are effectively automatic

Main theorem for FOX (Khoussainov, Rubin, Stephan '04)

From a presentation $P = (M, (M_i)_{1 \leq i \leq n})$ of a regular structure $\mathcal{A}$ and a FOX-formula α, one can compute a synchronous multitape-automaton M^α such that $R(M^\alpha) = \alpha^{\mathcal{A}}$.

Lemma

If $R \subseteq (\Sigma^*)^n$ is automatic and $p > 1$, then the set of tuples $\overline{u} \in (\Sigma^*)^{n-1}$ satisfying

$$|\{v \in \Sigma^* \mid (\overline{u}, v) \in R\}| \text{ is finite and divisible by } p$$

is effectively automatic.

Proof of Main Theorem for FOX

equals proof for FO □

Proof of Lemma for $n = 2$

(Σ^*, R) is automatic
$\Rightarrow R' = \{(u, v) \mid \exists^\infty w : (u, w) \in R\}$ effectively automatic

let $M = (Q, \{\iota\}, T, F)$ be deterministic finite automaton accepting $\otimes R'$

- $Q' = \{0, 1, \ldots, p-1\}^Q$
- $\iota'(p) = \begin{cases} 1 & \text{for } p = \iota \\ 0 & \text{otherwise} \end{cases}$
- $(f, a, g) \in T'$ iff $g(q) = \sum_{p \in Q} f(p) \cdot |\{b \in \Sigma \mid (p, (a, b), q) \in T\}| \bmod p$ for all $q \in Q$
- $f \in F'$ iff $\sum_{p \in Q} f(p) = 0 \bmod p$

Then $(Q', \{\iota'\}, T', F')$ accepts "something like" the set required. $\square$

FOX-definable relations are effectively automatic

Main theorem for FOX (Khoussainov, Rubin, Stephan '04)

From a presentation $P = (M, (M_i)_{1 \leq i \leq n})$ of a regular structure $\mathcal{A}$ and a FOX-formula α, one can compute a synchronous multitape-automaton M^α such that $R(M^\alpha) = \alpha^{\mathcal{A}}$.

Consequences

The FOX-theory of every automatic structure is decidable, automatic structures are closed under FOX-interpretations.

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Second-order logic

For a change

$(\mathbb{N} \times \mathbb{N}, \leq)$ is automatic and its second-order theory is undecidable.

Let $\mathcal{A} = (V, (R_i)_{1 \leq i \leq n})$ be a relational structure with $R_i \subseteq V^{k_i}$.

Syntax of our fragment FSO of second-order logic:

- formation rules for FOX
- if X is an n-ary relation variable and $x_1, \ldots, x_n$ are first-order variables, then $X(x_1, \ldots, x_n)$ is a formula.
- if X is a relation variable and α a formula s.t. $\forall Y, Z : \alpha(Y \cup Z) \to \alpha(Y)$ is a tautology, then $\exists X$ infinite : $\alpha(X)$ is a formula

free variables:

. . . . . .

Effectiveness results for FSO

Theorem (K, Lohrey '10)

(1) The set of pairs (P, α) with P a presentation of some regular structure $\mathcal{A}$ and α some sentence from FSO s.t. $\mathcal{A} \models \alpha$ is decidable.

(2) From a presentation P of some regular structure $\mathcal{A}$ and a sentence $\exists X$ infinite : α valid in $\mathcal{A}$, one can compute a synchronous multitape automaton M such that $R(M)$ is infinite and $(\mathcal{A}, R(M)) \models \alpha$.

Proof strategy for (1)

- second-order quantifications can be restricted to "combs"
- $\mathcal{A}$ together with all "combs" is an "ω-automatic structure"
- these ω-automatic structures share all the nice properties of automatic structures that we learnt to love

Combs

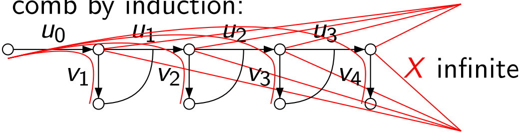
comb: set $\{u_0 u_1 u_2 \dots u_{n-1} v_n \mid n \in \mathbb{N}\} \subseteq \Sigma^+$ with $|v_i| < |u_i|$ f.a. $i \in \mathbb{N}$

Lemma

$X \subseteq \Sigma^+$ infinite, Σ finite $\Longrightarrow \exists C \subseteq X$ comb

Proof

build comb by induction:



2nd order quantification over combs

Consequence

2nd-order quantification in FSO-sentences can be restricted to combs.

Proof

let $\exists X$ infinite : α be formula from FSO

then $\forall C, R : C \subseteq R \wedge \alpha(R) \rightarrow \alpha(C)$ is a tautology

hence: $\mathcal{A} \models \exists X$ infinite : α

$\iff$ there is an infinite set R s.t. $(\mathcal{A}, R) \models \alpha$

$\iff$ there is an infinite comb C s.t. $(\mathcal{A}, C) \models \alpha$ □

Interim result

from $\alpha \in \mathrm{FSO}$, we can construct $\overline{\alpha} \in \mathrm{FOX}$ s.t. $\mathcal{A} \models \alpha$ if and only if $\overline{\alpha}$ holds in

$\overline{\mathcal{A}} = \Big(V \cup$ set of combs, all relations of $\mathcal{A}$, $\Big)$

Combs as ω-words

coding of comb $C = \{u_1 u_2 \dots u_{n-1} v_n \mid n \in \mathbb{N}\}$: ω-word c over $(\Sigma \cup \{\#\})^2$ of form

$v_0\#\dots\#$	$v_1\#\dots\#$	$v_2\#\dots\#$	$v_3\#\dots\#$	$v_4\#\dots\#$	$\dots$
u_0	u_1	u_2	u_3	u_4	$\dots$

hence $\overline{\mathcal{A}}$ is "ω-automatic" and validity of $\overline{\alpha}$ is decidable by Blumensath '99 and Bárány, Kaiser, Rubin '08 □

Summary

Theorem (K, Lohrey '10)

(1) The set of pairs (P, α) with P a presentation of some regular structure $\mathcal{A}$ and α some sentence from FSO s.t. $\mathcal{A} \models \alpha$ is decidable.

(2) From a presentation P of some regular structure $\mathcal{A}$ and a sentence $\exists X$ infinite : α valid in P, one can compute a synchronous multitape automaton M such that $R(M)$ is infinite and $(\mathcal{A}, R(M)) \models \alpha$.

(3) The class of automatic structures is closed under FSO-interpretations.

See you tomorrow!