Automatic structures
Lecture 3: Complexity of first-order logic

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The problem

SA . . . all automatic presentations

For $\mathsf{C} \subseteq \mathsf{SA}$ and $L \subseteq \mathrm{FO}$:

$$\mathrm{MC}(\mathsf{C}, L) = \{(P, \varphi) \mid P \in \mathsf{C}, \varphi \in L \text{ sentence}, \mathcal{A}(P) \models \varphi\}$$

is the model checking problem for L and C.

Theorem

$\mathrm{MC}(\mathsf{SA}, \mathrm{FO})$ is decidable.

Question

But what is the complexity of this decision problem, what are the difficult and easy instances?

Disappointing example

The first-order theory of the complete binary tree (with prefix relation) is non-elementary, hence $\mathrm{MC}(\mathsf{SA}, \mathrm{FO})$ is non-elementary.

The problem

Quantifier alternation

Bounded degree

What makes our decision procedure slow?

Lemma

From synchronous n-tape automata M_1 and M_k, one can compute synchronous n-tape automata

- for $R(M_i) \cup R(M_2)$ in polynomial time,
- for $R(M_i) \cap R(M_2)$ in polynomial time,
- for the projection of $R(M_1)$ in polynomial time,
- for the cylindrification of $R(M_1)$ in polynomial time,
- for the complement of $R(M_1)$ in exponential time.

Lemma

Emptyness of an automatic relation can be decided using nondeterministic logarithmic space.

Answer

The maximal number of nested negations in the formula.

Some more notation

Convention

In the rest of this section, we allow conjunction $\wedge$ in FO-formulas.

Definition

$\Sigma_0 \subseteq \mathrm{FO}$ is the set of quantifier-free formulas
$B\Sigma_n \subseteq \mathrm{FO}$ is the set of Boolean combinations of formulas from Σ_n
$\Sigma_{n+1} \subseteq \mathrm{FO}$ is the closure of the set $B\Sigma_n$ by existential quantification, $\vee$, and $\wedge$

Observation

Using de Morgan's laws, any formula from Σ_{n+1} can be written with at most $n+1$ nested negations and without increasing the size of the formula.

A first “simple” case

$\exp_0(n) = n$ and $\exp_{k+1}(n) = 2^{\exp_k(n)}$

*k*EXSPACE is the set of problems that can be solved in space $\exp_k(n^{O(1)})$ (with kEXSPACE = PSPACE)
$\bigcup_{k\geq 0} k$EXSPACE is the set of elementary problems

Lemma

$\mathrm{MC}(\mathsf{SA}, \Sigma_{n+1}) \in n$EXSPACE for all $n \geq 0$.

Proof

$\varphi \in \Sigma_{n+1}$ with at most $n+1$ nested negations and P automatic presentation of $\mathcal{A}$
build M s.t. $R(M) = \varphi^{\mathcal{A}}$ of $(n+1)$-fold exponential size
decide emptyness of $R(M)$ in space logarithmic in $|M|$
since final decision can be done “on-the-fly”, we need not store the huge automaton M. $\square$

Can we do any better?

Lemma (K '09)

- Data complexity: For $n \geq 0$, there exists a sentence $\varphi_n \in \Sigma_{n+1}$ s.t. $\mathrm{MC}(\mathsf{SA}, \{\varphi_n\})$ is nEXSPACE-hard.
- Expression complexity: There exists an automatic presentation P s.t. $\mathrm{MC}(\{P\}, \Sigma_{n+1})$ is nEXSPACE-hard for all $n \geq 0$ (for $n \geq 2$, this follows from Streid '90).

Theorem (K '09)

$\mathrm{MC}(\mathsf{SA}, \Sigma_{n+1})$ is nEXSPACE-complete.

The problem

Quantifier alternation

Bounded degree

Some model theory

Definition

$\mathcal{A} = (V; (R_i)_{1 \le i \le n}$ some (fixed) relational structure.

- $E = \{(u, v) \in V^2 \mid \exists \overline{w} \in \bigcup_{1 \le i \le n} R_i : u, v$ appear in tuple $w\}$
  $G(\mathcal{A}) = (V, E)$ is the Gaifman graph of $\mathcal{A}$.
- for $u, v \in V$: $d(u, v)$ is minimal length of path from u to v in $G(\mathcal{A})$ (possibly ∞)
- for $u \in V$, $r \in \mathbb{N}$: $S(r, u) = \{v \in V \mid d(u, v) \le r\}$
- $\varphi \in \mathrm{FO}$: $\mathrm{qr}(\varphi)$ is nesting depth of quantifiers in φ

Gaifman's locality principle

Theorem (Gaifman '82, Keisler & Lotfallah '05)

$\mathcal{A}$ relational structure, $u_i, v_i \in \mathcal{A}$ for $1 \leq i \leq k$,
$\varphi(x_1, \ldots, x_k) \in \mathrm{FO}$ with $\mathrm{qr}(\varphi) \leq r$,

$$(\mathcal{A}{\restriction}(\bigcup_{i=1}^{k} S(2^{r+k-i}, u_i)), \overline{u}) \cong (\mathcal{A}{\restriction}(\bigcup_{i=1}^{k} S(2^{r+k-i}, v_i)), \overline{v})\,.$$

Then

$$(\mathcal{A}, \overline{u}) \models \varphi \iff (\mathcal{A}, \overline{v}) \models \varphi\,.$$

Potential spheres

A potential (r,k)-sphere is a tuple $(\mathcal{B}, b_1, \ldots, b_k)$ s.t.

- $\mathcal{B}$ is a structure with $b_1, \ldots, b_k \in \mathcal{B}$.
- For all $b \in \mathcal{B}$ there exists $1 \leq i \leq k$ such that $d(b_i, b) \leq 2^{r-i}$.

The potential (r,k)-sphere $(\mathcal{B}, b_1, \ldots, b_k)$ is realizable in the structure $\mathcal{A}$ if there are $a_1, \ldots, a_k \in \mathcal{A}$ s.t.

$$(\mathcal{A}{\upharpoonright}(\bigcup_{i=1}^{k} S(2^{r-i}, a_i)), a_1, \ldots, a_k) \cong (\mathcal{B}, b_1, \ldots, b_k)\,.$$

Formulas and spheres

$\varphi(y_1, \dots, y_k) \in \mathrm{FO}$ with $\mathrm{qr}(\varphi) \leq r$ and $\sigma = (\mathcal{B}, b_1, \dots, b_k)$ a potential $(r+k, k)$-sphere.
Define $\varphi_\sigma \in \{0, 1\}$ inductively:

- If $\varphi(y_1, \dots, y_k)$ is an atomic formula, then

$$\varphi_\sigma = \begin{cases} 1 & \text{if } \mathcal{B} \models \psi(b_1, \dots, b_k) \\ 0 & \text{if } \mathcal{B} \not\models \psi(b_1, \dots, b_k) \,. \end{cases}$$

- If $\varphi = \neg\alpha$, then $\varphi_\sigma = 1 - \alpha_\sigma$.
- If $\varphi = \alpha \vee \beta$, then $\varphi_\sigma = \max(\alpha_\sigma, \beta_\sigma)$.
- If $\varphi(y_1, \dots, y_k) = \exists y_{k+1} : \alpha(y_1, \dots, y_k, y_{k+1})$ then

$$\varphi_\sigma = \max\left\{ \alpha_{\sigma'} \;\middle|\; \begin{array}{l} \sigma' \text{ is a potential } (r+k, k+1)\text{-sphere} \\ \text{realizable in } \mathcal{A} \text{ and extending } \sigma \end{array} \right\} .$$

"Non-standard" evaluation of formulas

Theorem

$\mathcal{A}$ a structure with $a_1, \ldots, a_k \in \mathcal{A}$, $\varphi(y_1, \ldots, y_k) \in \mathrm{FO}$ with $\mathrm{qr}(\varphi) \leq r$, and σ a potential $(r+k, k)$-sphere with

$$(\mathcal{A}{\restriction}(\bigcup_{i=1}^{k} S(2^{r+k-i}, a_i)),\ a_1, \ldots, a_k) \cong \sigma\,.$$

Then $\mathcal{A} \models \varphi(a_1, \ldots, a_k) \iff \varphi_\sigma = 1$.

Corollary

$\mathcal{A}$ a structure, $\varphi \in \mathrm{FO}$ a sentence with $\mathrm{qr}(\varphi) \leq r$, and $\emptyset$ the potential $(r, 0)$-sphere.
Then $\mathcal{A} \models \varphi \iff \varphi_\emptyset = 1$.

Problems for computing $\varphi_\emptyset$ for $\mathcal{A}$ automatic

$\varphi(y_1, \ldots, y_k) \in \mathrm{FO}$ with $\mathrm{qr}(\varphi) \leq r$ and $\sigma = (\mathcal{B}, b_1, \ldots, b_k)$ a potential $(r+k, k)$-sphere.
Define $\varphi_\sigma \in \{0, 1\}$ inductively:

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Structures of bounded degree

Definition

- A graph $G = (V, E)$ has bounded degree if there exists $d \in \mathbb{N}$ such that any node has at most d neigbhours, the minimal such d is the degree of G.
- A relational structure has bounded degree (degree d, resp.) if its Gaifman graph has bounded degree (degree d, resp.).
- SAb $\subset$ SA is the set of automatic presentations of bounded degree.

Crucial property

If $\mathcal{A}$ is a structure of degree d, then it realizes at most $\exp_3((k + \log d + r)^{O(1)})$ potential (r, k)-spheres (of size $\exp_2((k + \log d + r)^{O(1)})$).

The class SAb

Lemma

Given $P \in \text{SA}$, one can decide in polynomial time whether $P \in \text{SAb}$ and if so, the degree of $\mathcal{A}(P)$ is at most $\exp_1(|P|^{O(1)})$.

Proof

let $\mathcal{A} = \mathcal{A}(P)$.
E is definable in $\mathcal{A}$ by a positive Σ_1-formula
hence E can be accepted by a synchronous 2-tape automaton of polynomial size
boundedness of automatic relations is decidable in polynomial size (Weber '90) and the degree is at most exponential $\square$

Hence

If $P \in \text{SAb}$, then $\mathcal{A}(P)$ realizes at most $\exp_3((k + |P| + r)^{O(1)})$ potential (r, k)-spheres (of size $\exp_2((k + |P| + r)^{O(1)})$) and this set is "efficiently" decidable.

Former problems for computing $\varphi_\emptyset$ for $\mathcal{A}$ automatic

$\varphi(y_1, \dots, y_k) \in \mathrm{FO}$ with $\mathrm{qr}(\varphi) \leq r$ and $\sigma = (\mathcal{B}, b_1, \dots, b_k)$ a potential $(r+k, k)$-sphere.
Define $\varphi_\sigma \in \{0, 1\}$ inductively:

- If $\varphi(y_1, \dots, y_k)$ is an atomic formula, then

$$\varphi_\sigma = \begin{cases} 1 & \text{if } \mathcal{B} \models \psi(b_1, \dots, b_k) \\ 0 & \text{otherwise.} \end{cases}$$

- If $\varphi = \neg\alpha$, then $\varphi_\sigma = 1 - \alpha_\sigma$.
- If $\varphi = \alpha \vee \beta$, then $\varphi_\sigma = \max(\alpha_\sigma, \beta_\sigma)$.
- If $\varphi(y_1, \dots, y_k) = \exists y_{k+1} : \alpha(y_1, \dots, y_k, y_{k+1})$ then

$$\varphi_\sigma = \max \left\{ \alpha_{\sigma'} \;\middle|\; \begin{array}{l} \sigma' \text{ is a potential } (r+k, k+1)\text{-sphere} \\ \text{realizable in } \mathcal{A} \text{ and extending } \sigma \end{array} \right\}.$$

Harvest

Lemma

From $P \in$ SAb and $\varphi \in \mathrm{FO}$ sentence, one can compute $\varphi_\emptyset$ in doubly exponential space.

Theorem (K, Lohrey '09)

- $\mathrm{MC}(\text{SAb}, \mathrm{FO}) \in$ 2EXSPACE
- there exists $P \in$ SAb such that $\mathrm{MC}(\{P\}, \mathrm{FO})$ is 2EXSPACE-hard.
- If $P \in$ SAb s.t. the number of realizable spheres grows polynomial with the radius, then $\mathrm{MC}(\{P\}, \mathrm{FO}) \in$ EXSPACE.
- there exists $P \in$ SAb s.t. the number of realizable spheres grows polynomial with the radius and $\mathrm{MC}(\{P\}, \mathrm{FO})$ is EXSPACE-hard.

Combination of "quantifier alternation" and "bounded degree"

Recall

- $\mathrm{MC}(\mathsf{SA}, \Sigma_{n+1}) \in n\mathsf{EXSPACE}$ for all $n \geq 0$
- $\mathrm{MC}(\mathsf{SAb}, \mathrm{FO}) \in 2\mathsf{EXSPACE}$

Conjecture

$\mathrm{MC}(\mathsf{SAb}, \Sigma_n) \in \mathsf{EXSPACE}$ for all $n \geq 0$.

See you tomorrow!