Automatic structures
Lecture 4: Some classical graph problems

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The problem

typical graph problems φ

- Hamiltonicity
- existence of an Eulerian path
- existence of an infinite clique

Question 1

How difficult is it to tell whether $\mathcal{A}(P)$ has property φ?

Question 2

Are there "simple" characterizations of the computable and the automatic graphs with property φ?

Arithmetic and analytical hierarchy

universe $\mathcal{U}$: all finitary objects (e.g. natural numbers, words, automata, finite sets ...)

relations: all decidable relations on $\mathcal{U}$

Σ^0_n: all relations defined by formulas of form $\exists\overline{x}_1\forall\overline{x}_2\dots\exists/\forall\overline{x}_n : R(\overline{x},\overline{x}_1,\overline{x}_2,\dots,\overline{x}_n)$

Π^0_n: all relations defined by formulas of form $\forall\overline{x}_1\exists\overline{x}_2\dots\forall/\exists\overline{x}_n : R(\overline{x},\overline{x}_1,\overline{x}_2,\dots,\overline{x}_n)$

Σ^1_1: all relations defined by formulas of form $\exists X_1,\dots,X_m : \varphi$ with φ first-order, X_i relation variable

Infinite cliques

- ...in computable graphs
- ...in automatic graphs
- Summary

Eulerian graphs

- ...in computable graphs
- ...in automatic graphs
- Summary

Hamiltonian graphs

- ...in computable graphs
- ...in automatic graphs
- Summary

Computable graphs

$C =$ all pairs (M_1, M_2) of TM that decide nodes and edges of a computable graph with an infinite clique

$(M_1, M_2) \in C$ iff

$$\begin{aligned}\mathcal{U} \models \exists X \subseteq \mathbb{N} \exists Y \subset X \exists f \subseteq X \times Y : \\ & \forall u \in X : u \in L(M_1) \wedge \\ & \forall u, v \in X : (u, v) \in L(M_2) \vee u = v \wedge \\ & \forall x \in X \exists y \in Y : (x, y) \in f \wedge \\ & \forall x \in X, y_1, y_2 \in Y : (x, y_1), (x, y_2) \in f \rightarrow y_1 = y_2\end{aligned}$$

$\Rightarrow$ existence of infinite clique for computable graphs belongs to Σ_1^1

Infinite cliques in computable graphs

Theorem (Kleene '43)

existence of an infinite clique in a computable graph is Σ_1^1-complete

Question 1

How difficult is it to tell whether the computable graph G has an infinite clique?

Answer

It is Σ_1^1-complete ("highly undecidable").

Question 2

Is there a "simple" characterization of the computable graphs with an infinite clique?

Answer

no since there is no better way of saying "the computable graph G contains an infinite clique" than the obvious one

Infinite cliques in automatic graphs

Theorem (Rubin '08)

existence of infinite clique in an automatic graph is decidable.

Proof

$\exists X$ infinite $\forall x, y : (x, y \in X \wedge x \neq y) \rightarrow (x, y) \in E$

- is formula of FSO,
- expresses "there is an infinite clique",
- and its validity in an automatic graph is decidable □

Question 1

How difficult is it to tell whether the automatic graph G has an infinite clique?

Answer

It is decidable.

Summary: infinite cliques

Question 1

How difficult is it to tell whether a computable and an automatic graph have an infinite clique?

Answer

highly undecidable vs. decidable

Question 2

Are there "simple" characterizations of the computable and the automatic graphs with infinite cliques?

Answer

no vs. yes

Infinite cliques

. . . in computable graphs

. . . in automatic graphs

Summary

Eulerian graphs

. . . in computable graphs

. . . in automatic graphs

Summary

Hamiltonian graphs

. . . in computable graphs

. . . in automatic graphs

Summary

Eulerian paths

Eulerian path: bijection $f : \{n \in \mathbb{N} \mid n < |E|\} \to E$ such that $f(n)$ and $f(n+1)$ have one node in common and $f(n)$, $f(n+1)$, $f(n+2)$ do not (for all n)

Theorem (Euler 1736)

$G = (V, E)$ finite. Then G has an Eulerian path if and only if

1. G has at most one nontrivial connected component
2. G has at most two nodes of odd order

Eulerian computable graphs

$E =$ all pairs (M_1, M_2) of TM that decide nodes and edges of an Eulerian graph $(L(M_1), L(M_2))$

$(M_1, M_2) \in E$ iff

$$\mathcal{U} \models \exists f : f : \mathbb{N} \to L(M_2) \text{ bijection} \wedge$$
$$\forall n \in \mathbb{N} \exists v \in L(M_1) : v \quad \text{belongs to } f(n) \text{ and to } f(n+1), \text{ but not to } f(n+2)$$

$\Longrightarrow$ existence of an Eulerian path in computable graphs belongs to Σ^1_1

Eulerian computable graphs

Erdős, Grünwald & Vazsonyi 1938

$G = (V, E)$ countable. Then G has an Eulerian path if and only if

1. G has at most one nontrivial connected component: in Π^0_2
   $\forall (x, x'), (y, y') \in E \, \exists$ path from x to y
2. G has a vertex of odd or infinite order: in Σ^0_3
   $\exists x \in V, M \subseteq_{fin} V : \forall y (x \, E \, y \leftrightarrow y \in M) \wedge |M|$ odd
   $\vee \exists x \in V \forall M \subseteq_{fin} V \exists y : y \notin M \wedge x \, E \, y$
3. G has at most one vertex of odd order: in Π^0_3
   $\forall x, y \in V \forall M, N \subseteq_{fin} V : \ldots \to x = y$
4. G has only one end (i.e., removing finitely many edges leaves only one infinite connected component) G has only one end: in Π^0_3
   $\forall M \subseteq_{fin} E \forall x, y \in V :$
   $\exists$ path from x to y in $(V, E \setminus M) \vee$
   $\exists Z \subseteq_{fin} V : \ (x \in Z \vee y \in Z) \wedge$
   $\forall (z, z') \in E \setminus M : (z \in Z \leftrightarrow z' \in Z)$

Eulerian computable graphs

Theorem (K & Lohrey '10)

existence of an Eulerian path in computable graphs is complete for $D^0_3 = \{K \cap L \mid K \in \Sigma^0_3, L \in \Pi^0_3\}$.

Question 1

How difficult is it to tell whether a computable graph is Eulerian?

Answer

It is D^0_3-complete.

Question 2

Is there a "simple" characterization of the computable Eulerian graphs?

Answer

yes, and the characterization by Erdős, Grünwald & Vazsonyi is optimal.

Eulerian automatic graphs

Erdős, Grünwald & Vazsonyi 1938

$G = (V, E)$ countable. Then G has an Eulerian path if and only if

1. G has at most one nontrivial connected component: in Π^0_2
   $\forall (x, x'), (y, y') \in E \; \exists$ path from x to y
2. G has a vertex of odd or infinite order: decidable
   $\exists x \neg \exists^{(2)} y : (x, y) \in E$
3. G has at most one vertex of odd order: decidable
   $\exists x \forall y : x \neq y \to \exists^{(2)} z : (y, z) \in E \vee \exists^{\infty} z : (y, z) \in E$
4. G has only one end: in Π^0_2
   $\forall M \subseteq_{fin} E \forall x, y \in V :$
   $\quad \exists$ path from x to y in $(V, E \setminus M) \vee$
   $\quad \exists Z \subseteq_{fin} V : \; (x \in Z \vee y \in Z) \wedge$
   $$\underbrace{\forall (z, z') \in E \setminus M : (z \in Z \leftrightarrow z' \in Z)}_{\text{decidable since } (V; E, M, Z) \text{ automatic}}$$

Eulerian automatic graphs

Theorem (K & Lohrey '10)

existence of an Eulerian path in automatic graphs is Π^0_2-complete

Question 1

How difficult is it to tell whether an automatic graph is Eulerian?

Answer

It is Π^0_2-complete.

Question 2

Is there a "simple" characterization of the automatic Eulerian graphs?

Answer

yes, and the characterization by Erdős, Grünwald & Vazsonyi together with decidability of logic FOX leads to optimal characterization

Summary: Eulerian path

Question 1

How difficult is it to tell whether a computable and an automatic graph have an Eulerian path?

Answer

D^0_3- vs. Π^0_2-complete

Question 2

Are there "simple" characterizations of the computable and the automatic graphs with an Eulerian path?

Answer

yes in both cases (slightly "simpler" for automatic graphs)

Hamiltonian computable graphs

Hamiltonian path: bijection $f : \{n \in \mathbb{N} \mid n < |V|\} \to V$ such that $(f(n), f(n+1)) \in E$ for all n

$H =$ all pairs (M_1, M_2) of TM that decide nodes and edges of a computable Hamiltonian graph $(L(M_1), L(M_2))$

$(M_1, M_2) \in H$ iff

$$\begin{aligned}\mathcal{U} \models \exists f \subseteq \mathbb{N}^2\ & \forall n \in \mathbb{N} \exists v \in L(M_1) : (n, v) \in f\ \wedge \\ & \forall v \in L(M_1) \exists n \in \mathbb{N} : (n, v) \in f\ \wedge \\ & \forall n, v_1, v_2 : (n, v_1), (n, v_2) \in f \to v_1 = v_2\ \wedge \\ & \forall n_1, n_2, v : (n_1, v), (n_2, v) \in f \to n_1 = n_2\ \wedge \\ & \forall n \in \mathbb{N} : (f(n), f(n+1)) \in L(M_2)\end{aligned}$$

$\Longrightarrow$ Hamiltonicity for computable graphs belongs to Σ^1_1

Hamiltonian computable graphs

Theorem (Hirst & Harel '96)

Hamiltonicity for computable graphs is Σ^1_1-complete.
(hardness even for planar graphs and for graphs of bounded degree)

hardness proof

reduction from Kleene's Σ^1_1-complete problem "computable tree has infinite branch"

Question 1

How difficult is it to tell whether a computable graph is Hamiltonian?

Answer

It is Σ^1_1-complete.

Question 2

Infinite cliques
- . . . in computable graphs
- . . . in automatic graphs
- Summary

Eulerian graphs
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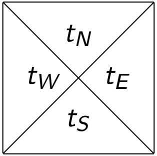
- . . . in automatic graphs
- Summary

Hamiltonian graphs
- . . . in computable graphs
- . . . in automatic graphs
- Summary

Recurring tilings

tiling system: tuple $(C, \mathcal{T}, c_0)$ with $C = \{c_0, c_1, \ldots, c_n\}$ finite set of colors, $\mathcal{T} \subseteq C^4$

tile $(t_W, t_N, t_E, t_S) \in C^4$ visualised as:



recurring tiling problem:
all tiling systems that allow tiling of $\mathbb{N} \times \mathbb{N}$ such that color c_0 occurs infinitely often at bottom border

Theorem (Harel '91)

Recurring tiling problem is Σ^1_1-complete

Hamiltonian automatic graphs

Theorem (K & Lohrey '10)

There is a constant c such that, from a tiling system $\mathcal{T} = (C, \mathcal{T}, c_0)$, one can compute a planar automatic graph $G_{\mathcal{T}}$ of degree c such that the following are equivalent

- $\mathcal{T}$ allows a recurring tiling of $\mathbb{N}^2$
- $G_{\mathcal{T}}$ is Hamiltonian.

Hence: Hamiltonicity of automatic planar graphs of degree c is Σ^1_1-complete

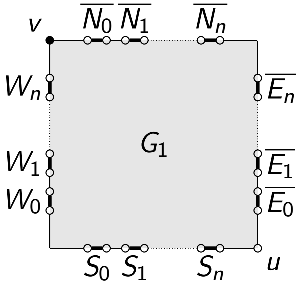
1st step: coding of $\mathcal{T}$ in finite graphs (1)

define $\mathcal{V} = \{W_i, S_i, \overline{N_i}, \overline{E_i} \mid 0 \leq i \leq n\}$

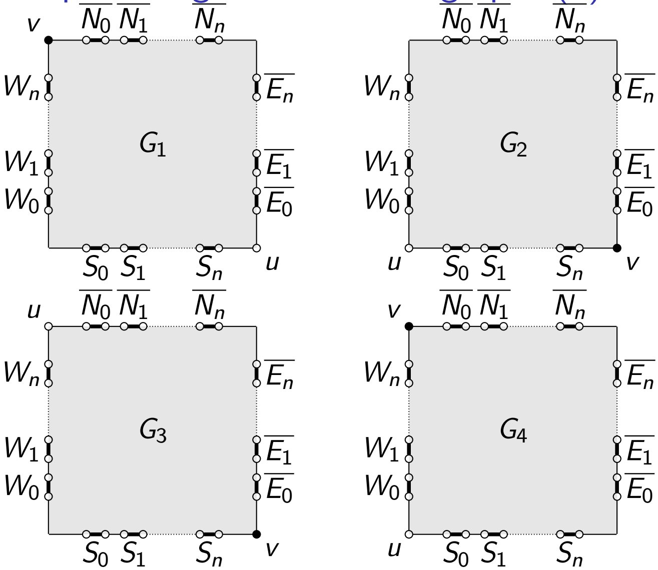
Lemma (minor extension of Garey, Johnson & Tarjan '76)

$\exists$ finite planar graphs $G_1, \ldots, G_4$ s.t. $\forall t = (c_i, c_j, c_k, c_\ell) \in C^4$:
$t \in \mathcal{T}$ iff $\exists$ Hamiltonian path P from u to v s.t.

- $W_m \in P \iff m = i$, $S_m \in P \iff m = j$
- $\overline{N}_m \in P \iff m \neq k$, and $\overline{E}_m \in P \iff m \neq \ell$

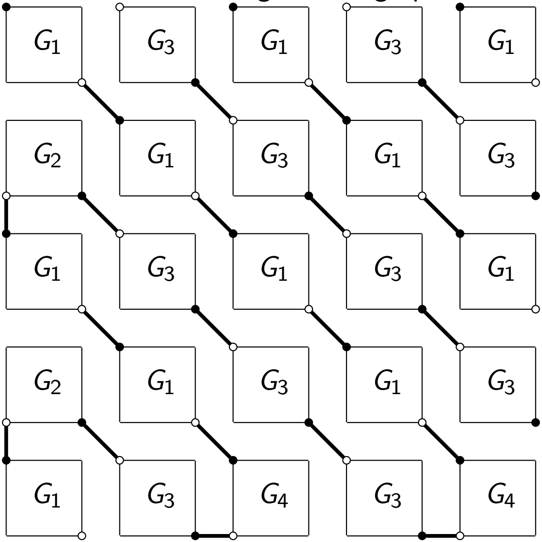


1st step: coding of $\mathcal{T}$ in finite graphs (2)



2nd step: placing graphs in plane

Let G^1 denote the following infinite graph:



Interim result after 2 steps

u-rooted graph G^1 and correspondence between

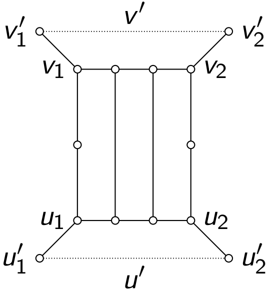
1. Hamiltonian paths H from u
2. mappings $\mathbb{N}^2 \to \mathcal{T}$

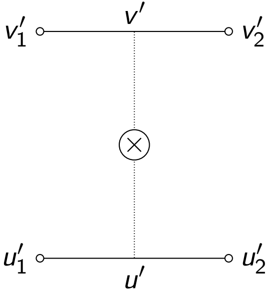
next problem: extend G^1 to G^2 s.t. for all $i, j \in \mathbb{N}$ and all Hamiltonian paths H of G^2:

$$W_k(i+1, j) \in H \text{ iff } \overline{E}_k(i, j) \notin H$$
$$S_k(i, j+1) \in H \text{ iff } \overline{N}_k(i, j) \notin H$$

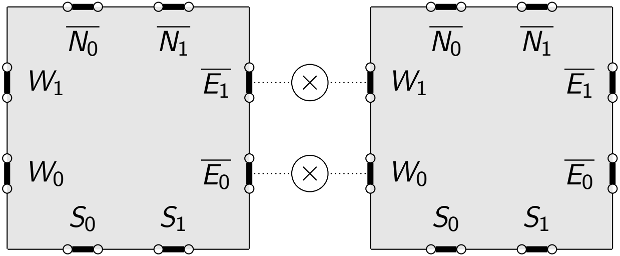
3rd step: synchronisation of colors (1)

exclusive or ("folklore")





3rd step: synchronisation of colors (2)



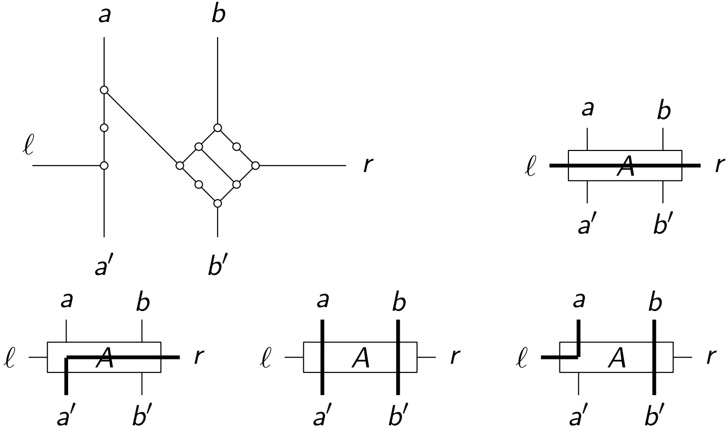
Interim result after 3 steps

u-rooted graph G^2 and correspondence between

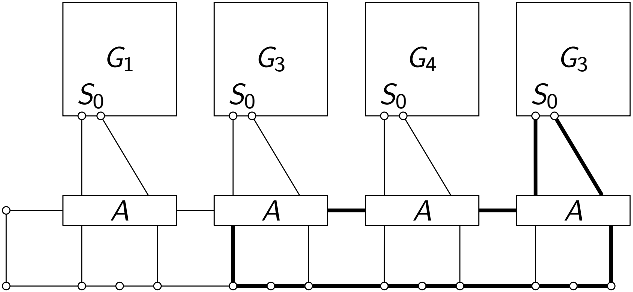
1. Hamiltonian paths H from u
2. tilings of $\mathbb{N}^2$ with tiles from $\mathcal{T}$

remaining problem: color c_0 shall occur infinitely often on bottom border

4st step: recurrence checking (1)



4st step: recurrence checking (2)



Interim result after 4 steps

u-rooted graph G^3 and correspondence between

1. Hamiltonian paths H from u
2. tilings of $\mathbb{N}^2$ with tiles from $\mathcal{T}$ s.t. color c_0 appears infinitely often at bottom border

last remaining problem: automaticity of G^3
holds since G^3 can be FOX-interpreted in grid $(\mathbb{N} \times \mathbb{N}, \leq)$
hence: Hamiltonicity of automatic planar graphs of degree c is Σ^1_1-complete

Hamiltonian automatic graphs

Theorem (K & Lohrey '10)

Hamiltonicity for automatic planar graphs of bounded degree is Σ^1_1-complete.

Question 1

How difficult is it to tell whether an automatic graph is Hamiltonian?

Answer

It is Σ^1_1-complete.

Question 2

Is there a "simple" characterization of the automatic Hamiltonian graphs?

Answer

no.

Summary: Hamiltonian path

Question 1

How difficult is it to tell whether a computable and an automatic graph have a Hamiltonian path?

Answer

Σ^1_1-complete in both cases

Question 2

Are there "simple" characterizations of the computable and the automatic graphs with a Hamiltonian path?

Answer

no in both cases

Summary

there exist (natural) problems φ s.t.

1. φ(automatic) decidable, φ(computable) Σ^1_1-complete (e.g., infinite clique)
2. φ(automatic) Π^0_2-complete, φ(computable) D^0_3-complete (e.g., Euler path, 1-endedness)
3. φ(automatic) and φ(computable) Σ^1_1-complete (e.g., Hamiltonicity, existence of infinite path)

open:

1. explanation for difference between 1, 2, and 3
2. investigation of subclasses of automatic structures

See you tomorrow!