Automatic structures
Lecture 5: Classification and isomorphism

Dietrich Kuske

LaBRI, Université de Bordeaux and CNRS

The classification problem

1. given an automatic presentation P, it is decidable whether $\mathcal{A}(P)$ is a linear order
2. since we can effectively list all automatic presentations, we can therefore effectively list all automatic linear orders (via their presentations)
3. this list contains repetitions – can they be avoided effectively?

A classification is a list of all elements of a class without repetition of isomorphic structures

General problem

Can we find "simple" classification of, e.g., automatic linear orders?

Ordinals

- Classification
- Isomorphism

The classification problem

- Equivalence structures
- Other classses of transitive structures

Summary

Definition and examples

An ordinal is a linear order $(V; \leq)$ not embedding $(\{\dots, -3, -2, -1, 0\}, \leq)$

Examples

- all finite linear orders $\mathbf{n} = (\{0, 1, \dots, n-1\}, \leq)$
- $\omega = (\mathbb{N}, \leq)$
- $\omega + \mathbf{1} = (\mathbb{N} \cup \{\infty\}, \leq)$, but $\mathbf{1} + \omega = \omega$
- $\omega \cdot \mathbf{2} = (\mathbb{N} \times \{0, 1\}, \leq_{\mathrm{lex}}) \cong \omega + \omega$
- $\omega^n = (\mathbb{N}^n, \leq_{\mathrm{lex}})$ for $n \in \mathbb{N}$ (with $\omega^0 = \mathbf{1}$)
- $\omega^\omega = \omega^0 + \omega^1 + \omega^2 + \omega^3 \cdots = (\mathbb{N}^+, \leq_{\mathrm{llex}})$ is least ordinal larger than any ω^n for $n \in \mathbb{N}$

Cantor's normal form

Fact

If $\alpha \in \mathbb{N}$, then there exists a unique tuple of natural numbers $(k_0, k_1, \ldots, k_n)$ with $0 \leq k_0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$ such that

$$\alpha = 10^{k_n} + \cdots + 10^{k_2} + 10^{k_1} + 10^{k_0}\,.$$

Lemma

If $\alpha < \omega^\omega$ is an ordinal, then there exists a unique tuple of natural numbers $(k_0, k_1, \ldots, k_n)$ with $0 \leq k_0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$ such that

$$\alpha = \omega^{k_n} + \cdots + \omega^{k_2} + \omega^{k_1} + \omega^{k_0}\,.$$

Automaticity of ordinals

Lemma

Ordinals $\alpha < \omega^\omega$ have the form $\alpha = \omega^{k_n} + \cdots + \omega^{k_2} + \omega^{k_1} + \omega^{k_0}$ with $0 \leq k_0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$.

Lemma

Any finite ordinal $\mathbf{n}$ is automatic.

Lemma

Any ordinal $\omega^n \cong ((0^*1)^n, \leq_{\mathrm{lex}})$ for $n \geq 1$ is automatic.

Lemma

If $\alpha = (A, \leq_A)$ and $\beta = (B, \leq_B)$ are automatic ordinals, then $\alpha + \beta \cong (A \uplus B, \leq_A \cup \leq_B \cup (A \times B))$ is an automatic ordinal.

Consequence

Any ordinal $\alpha < \omega^\omega$ is automatic and from $0 \leq k_0 \leq \cdots \leq k_n$, one can compute an automatic presentation of $\omega^{k_n} + \cdots + \omega^{k_1} + \omega^{k_0}$.

Automatic ordinals

Theorem (Delhommé, Goranko & Knapik '03)

An ordinal α is automatic if and only if $\alpha < \omega^\omega$

Since Cantor's normal form is unique, this gives an effective list of all automatic ordinals without repetitions:

- list all tuples $k_0 \geq k_1 \geq k_2 \geq \cdots \geq k_0 \geq 0$ of any length
- from an entry $\overline{k}$ in this list, compute an automatic presentation of $\omega^{k_n} + \cdots + \omega^{k_1} + \omega^{k_0}$.

Hence: There exists a computable classification of all automatic ordinals.

Ordinals

The isomorphism problem

P and P' automatic presentations of ordinals
$\alpha = \omega^{k_n} + \cdots + \omega^{k_1} + \omega^{k_0}$ and $\beta = \omega^{\ell_m} + \cdots + \omega^{\ell_1} + \omega^{\ell_0}$

$\alpha \cong \beta$ if and only if $\overline{k} = \overline{\ell}$ (since Cantor's normal form is unique).

to decide whether $\alpha \cong \beta$, it suffices to compute $\overline{k}$ and $\overline{\ell}$.

Computation of the sequence $\overline{k}$

$h := 0$; $\overline{k} := ()$
while $\mathcal{A}(P) \neq \mathbf{0}$
    $\overline{k} := (0, \overline{k})$
    while $\mathcal{A}(P)$ has maximal element
        compute automatic presentation P' with $\mathcal{A}(P) \cong \mathcal{A}(P') + 1$
        $P := P'$
        $\overline{k} = (h, \overline{k})$
    compute automatic presentation P' with $\mathcal{A}(P) \cong \mathcal{A}(P') \cdot \omega$
    $h := h + 1$
    $P := P'$

The isomorphism problem for ordinals

Theorem (Khoussainov, Nies, Rubin, Stephan '04)

The isomorphism problem for automatic ordinals is decidable (but no primitive recursive procedure is known).

Morale

A "good" classification leads to a "simple" isomorphism problem.

Similar story for automatic Boolean algebras, fields, and f.g. groups

but no further classifications are known!

Non-classifyability?

Theorem (Goncharov & Knight '02)

$\mathcal{C}$ hyperarithmetic[1] class of computable structures.

“simple” automatic

1. A hyperarithmetic isomorphism problem implies a hyperarithmetic classification.
   “simple”

“simple”

2. A Σ_1^1-complete isomorphism problem implies the non-existence of a hyperarithmetic classification.
   “difficult” hints at
   “simple”

Theorem (Khoussainov, Nies, Rubin & Stephan '07)

The isomorphism problem for automatic successor trees is Σ_1^1-complete – hence a “simple” classification of all automatic successor trees is unlikely.

[1] A set L is hyperarithmetic if L and its complement belong to Σ_1^1.

The isomorphism problem

For a class of structures $\mathfrak{C}$, what is the complexity of the set of pairs (P, P') of automatic presentations with $\mathcal{A}(P) \cong \mathcal{A}(P') \in \mathfrak{C}$?

Theorem (Khoussainov, Nies, Rubin, Stephan '07)

The isomorphism problem for automatic successor trees is Σ^1_1-complete.

Theorem (Rubin '04)

For automatic locally finite directed graphs, the isomorphism problem is Π^0_3-complete.

Theorem (Khoussainov, Nies, Rubin, Stephan '04)

The isomorphism problems for automatic ordinals and automatic Boolean algebras are decidable.

The isomorphism problem for further classes

- Rubin '08:
  Is isomorphism problem for equivalence structures decidable?
  - isomorphism of automatic equivalence structures decidable if all equivalence classes contain at most *n* elements (Khoussainov & Nerode '95)
  - $\exists$ automatic equivalence structure s.t. all equivalence classes finite, but arbitrarily large (Khoussainov & Nerode '95)
  - isomorphism problem for equivalence structures is in Π^0_1 (Rubin '08)
- we '09:
  Is isomorphism problem for several classes of trees decidable?
- Khoussainov, Rubin & Stephan '03:
  Is isomorphism problem for linear orders decidable?
  - decidable for ordinals and for FC-rank 0
  - automatic linear orders have finite FC-rank (Khoussainov, Rubin & Stephan '03)

Aim and strategy

Theorem (K, Liu & Lohrey '10)

The isomorphism problem for automatic equivalence structures is Π^0_1-complete.

Strategy

(0) build an equivalence structure $\mathcal{E}_{\mathrm{Good}}$ (the "good structure")

(1) from polynomials $p_1, p_2 \in \mathbb{N}[\overline{x}]$, build automatic equivalence structure $\mathcal{E}_{p_1,p_2}$ (the "test structure") with

$$\mathcal{E}_{p_1,p_2} \cong \mathcal{E}_{\mathrm{Good}} \iff \forall \overline{c} : p_1(\overline{c}) \neq p_2(\overline{c})$$

(2) use Matiyasevitch: $\{(p_1, p_2) \in \mathbb{N}[\overline{x}] \mid \forall \overline{c} : p_1(\overline{c}) \neq p_2(\overline{c})\}$ is Π^0_1-hard

The "good structure"

For a countable equivalence structure $\mathcal{E}$, define

$$\begin{array}{rcl} h_{\mathcal{E}} : \; (\mathbb{N}_{>0} \cup \{\infty\}) & \to & \mathbb{N} \cup \{\infty\} \\ h_{\mathcal{E}}(x) & \mapsto & \#\text{ equivalence classes of } \mathcal{E} \text{ of size } x \end{array}$$

The function $h_{\mathcal{E}}$ describes $\mathcal{E}$ up to isomorphism.

$C : \mathbb{N} \times \mathbb{N} \to \mathbb{N}_{>0} : (x, y) \mapsto (x + y)^2 + 3x + y + 1$ is injective

$\mathcal{E}_{\mathrm{Good}}$ is countably infinite equivalence structure s.t.

$$h_{\mathcal{E}_{\mathrm{Good}}}(n) = \begin{cases} \infty & \text{if } \exists x, y \in \mathbb{N} : x \neq y \text{ and } n = C(x, y) \\ 0 & \text{otherwise} \end{cases}$$

i.e., $\mathcal{E}_{\mathrm{Good}}$ "encodes" $\{(x, y) \mid x \neq y\}$

The "test structure"

Let $p_1, p_2 \in \mathbb{N}[x_1, \ldots, x_k]$. Consider the following polynomials

- $S_1(\overline{x}) = C(p_1(\overline{x}), p_2(\overline{x}))$
- $S_2(\overline{x}) = C(x_1, x_1 + x_2 + 1)$
- $S_3(\overline{x}) = C(x_1 + x_2 + 1, x_1)$

folklore

There are nondeterministic finite automata $\mathcal{A}_i$ s.t.

- $L(\mathcal{A}_i) \subseteq \alpha_1^* \alpha_2^* \ldots \alpha_k^*$
- $\forall \overline{c} \in \mathbb{N}^k$: $\mathcal{A}_i$ has precisely $S_i(\overline{c})$ accepting runs on $\alpha^{\overline{c}} := \alpha_1^{c_1} \alpha_2^{c_2} \ldots \alpha_k^{c_k}$.

The "test structure" – continued

Let V_i denote the set of accepting runs of $\mathcal{A}_i$ and $\rho \sim_i \sigma$ iff ρ and σ are runs on the same word $u \in \alpha_1^* \alpha_2^* \dots \alpha_k^*$.

$\mathcal{E}_i = (V_i, \sim_i)$ is automatic equivalence structure s.t.

$$\begin{aligned} h_{\mathcal{E}_i}(n) > 0 \quad & \text{iff} \quad \exists \text{ equivalence class with } n \text{ elements} \\ & \text{iff} \quad \exists \overline{c} : \mathcal{A}_i \text{ has precisely } n \text{ runs on } \alpha^{\overline{c}} \\ & \text{iff} \quad n \in \mathrm{Im}(S_i) \end{aligned}$$

$\mathcal{E}_i^\omega = (V_i\$^*, \equiv_i)$ with $\rho\$^m \equiv_i \sigma\$^n$ iff $\rho \sim_i \sigma$ and $m = n$ is automatic equivalence structure s.t.

$$h_{\mathcal{E}_i^\omega}(n) = \begin{cases} \infty & \text{if } n \in \mathrm{Im}(S_i) \\ 0 & \text{otherwise} \end{cases}$$

The "test structure" – continued

$$h_{\mathcal{E}_i^\omega}(n) = \begin{cases} \infty & \text{if } n \in \mathrm{Im}(S_i) \\ 0 & \text{otherwise} \end{cases}$$

i.e., $\mathcal{E}_1^\omega$ encodes $\{(p_1(\overline{c}), p_2(\overline{c})) \mid \overline{c} \in \mathbb{N}^k\}$, $\mathcal{E}_2^\omega$ encodes $\{(x,y) \mid x < y\}$, and $\mathcal{E}_3^\omega$ encodes $\{(x,y) \mid x > y\}$

$\mathcal{E}_{p_1,p_2} = \mathcal{E}_1^\omega \uplus \mathcal{E}_2^\omega \uplus \mathcal{E}_3^\omega$ is automatic equivalence structure s.t.

$$h_{\mathcal{E}_{p_1,p_2}}(n) = \begin{cases} \infty & \text{if } n \in \bigcup_{1 \le i \le 3} \mathrm{Im}(S_i) \\ 0 & \text{otherwise} \end{cases}$$

i.e. $\mathcal{E}_{p_1,p_2}$ encodes $\{(p_1(\overline{c}), p_2(\overline{c})) \mid \overline{c} \in \mathbb{N}^k\} \cup \{(x,y) \mid x \neq y\}$

Comparision of "good" and "test structure"

$$
\begin{aligned}
\mathcal{E}_{p_1,p_2} \cong \mathcal{E}_{\mathrm{Good}} &\iff \mathcal{E}_{p_1,p_2} \text{ and } \mathcal{E}_{\mathrm{Good}} \text{ encode the same sets} \\
&\iff \{(p_1(\overline{c}), p_2(\overline{c})) \mid \overline{c} \in \mathbb{N}^k\} \subseteq \{(x,y) \mid x \neq y\} \\
&\iff \forall \overline{c} : p_1(\overline{c}) \neq p_2(\overline{c})
\end{aligned}
$$

i.e. we proved

Theorem (K, Liu & Lohrey '10)

It is undecidable whether an automatic equivalence structure $\mathcal{E}$ with $h_{\mathcal{E}}(\infty) = 0$ and $\mathrm{Im}(h_{\mathcal{E}}) = \{0, \infty\}$ is isomorphic to $\mathcal{E}_{\mathrm{Good}}$.

Corollary (K, Lohrey & Jiu '10)

The isomorphism problem of automatic equivalence structures is Π^0_1-complete.

Ordinals

Classification

Isomorphism

The classification problem

Equivalence structures

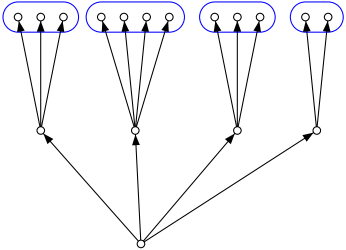
Other classses of transitive structures

Summary

Equivalence structures as trees of height 2

$\mathcal{E} =$

$T_{\mathcal{E}} =$



- $\mathcal{E} \cong \mathcal{E}' \iff T_{\mathcal{E}} \cong T_{\mathcal{E}'}$
- $T_{\mathcal{E}}$ FO-interpretable in $(\mathcal{E}, \leq_{\text{llex}})$ and hence effectively automatic

Consequence (K, Lohrey & Jiu '10)

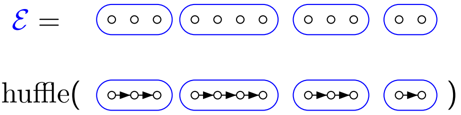
The isomorphism problem for automatic trees of height ≤ 2 is Π^0_1-hard.

The isomorphism problem for order trees

Theorem (K, Liu & Lohrey '10, '11)

1. The isomorphism problem for automatic trees of height ≤ 1 is decidable.
2. For $n \geq 2$, the isomorphism problem for automatic trees of height $\leq n$ is Π^0_{2n-3}-complete.
3. The isomorphism problem for automatic trees of finite height is Δ^0_ω-complete (i.e., equivalent to true arithmetic).
4. The isomorphism problem for automatic well-founded trees is in Δ^0_ω.
5. The isomorphism problem for automatic trees is Σ^1_1-complete.

Equivalence structures as linear orders



- if $h_{\mathcal{E}}(n), h_{\mathcal{E}'}(n) \in \{0, \infty\}$ for all $n \in \mathbb{N}_{>0} \cup \{\infty\}$:
  $\mathcal{E} \cong \mathcal{E}' \iff L_{\mathcal{E}} \cong L_{\mathcal{E}'}$
- $L_{\mathcal{E}}$ is effectively automatic

Consequence (K, Lohrey & Jiu '10)

The isomorphism problem for automatic linear orders is Π^0_1-hard.

The isomorphism problem for linear orders

Theorem (K, Liu & Lohrey '10, '11)

1. The isomorphism problem for automatic linear orders is Σ^1_1-complete., and this holds even for linear orders of FC rank 1.
2. The isomorphism problem for automatic scattered linear orders is in Δ^0_ω.

Summary

There are hyperarithmetic classifications of automatic

"simple"

(a) ordinals, Boolean algebras, equivalence structures,

(b) trees of bounded (or finite) height,

(c) well-founded trees, and

(d) scattered linear orders

since the isomorphism problems are hyperarithmetic.

There seem to be no hyperarithmetic "simple" classification of automatic

"simple"

(e) trees and

(f) linear orders

since the isomorphism problems are not hyperarithmetic.

It won't be easy to find hyperarithmetic "difficult" classifications in cases (b-d), since the isomorphism "simple" problems are not that simple.

Challenge

find a useful classification of at least one of the above classes

See you ~~tomorrow~~ at the reception!