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Infinite series-parallel posets: logic and languages

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Abstract. We show that a set of uniformly width-bounded infinite series-parallel pomsets is ω-series-rational iff it is axiomatizable in monadic second order logic iff it is ω-recognizable. This extends recent work by Lodaya and Weil on sets of finite series-parallel pomsets in two aspects: It relates their notion of series-rationality to logical concepts, and it generalizes the equivalence of recognizability and series-rationality to infinite series-parallel pomsets.

1 Introduction

In theoretical computer science, finite words are a classical concept that is used to model the behavior of a sequential system. In this setting, the atomic actions of the system are considered as letters of an alphabet Γ. A natural operation on such sequential behaviors is the concatenation; it models that, after finishing one task, a system can start another one. Therefore, the natural mathematical model is that of a (free) monoid $\Gamma^\star$. To model not only the behavior of a sequential system, but also allow parallelism, labeled partially ordered sets or pomsets were suggested [11, 18, 10]. In this setting, there is not only one, but there are (at least) two natural operations: A parallel system can start a new job after finishing the first one, or it can perform two jobs in parallel. These two operations are mathematically modeled by the sequential and the parallel product on pomsets: In the sequential product, the second pomset is set on top of the first. Complementary, in the parallel product, the two pomsets are put side by side. Thus, in the sequential product all events of the first factor are related to all events of the second while in the parallel product no additional relations are inserted. Another approach is that of Mazurkiewicz traces. Here, the sequentiality/parallelism is dictated by a fixed dependence relation on the set of actions. Therefore, the trace product (w.r.t. a given dependence relation) of two pomsets relates only dependent events of the two factors.

Pomsets that one obtains by the sequential and the parallel product from the singletons are known as series-parallel pomsets. It was shown that finite series-parallel pomsets are precisely those pomsets that do not contain a subposet of the form N (hence their alternative name "N-free posets") [10]. Together with the sequential and the parallel product, the finite N-free pomsets form an algebra, called sp-algebra, that generalizes the free monoid $\Gamma^\star$. The equational theory of this algebra was considered in [10]. Pomsets constructed from the singletons by the trace product are called traces. Together with the trace product, they form a monoid, called trace monoid. See [4] for a recent survey on the many results known on traces.

Several models of computational devices are known in theoretical computer science. The probably simplest one is that of a finite automaton, i.e. a finite state device capable of accepting or rejecting words. Several characterizations of the accepting power of finite automata are known: A set of words L can be accepted by a finite automaton if it is rational (Kleene), axiomatizable in monadic second order logic

(Büchi) or recognizable by a homomorphism into a finite monoid (Myhill-Nerode). Several attempts have been made to resume the success story of finite automata to pomsets, i.e. to transfer the nice results from the setting of a sequential machine to concurrent systems. For traces, this was achieved to a large extend by asynchronous (cellular) automata [22] (see [5] for an extension to pomsets without autoconcurrency). For N-free pomsets, Lodaya and Weil introduced branching automata. In [15, 16] they were able to show that a set of finite width-bounded N-free pomsets is rational iff series-rational (i.e. can be constructed from the singletons by union, sequential and parallel product and by the sequential iteration) iff recognizable (i.e. saturated by a homomorphism into a finite sp-algebra). This was further extended in [14] by the consideration of sets that are not uniformly width-bounded.

While finite words are useful to deal with the behavior of terminating systems, ω-words serve as a model for the behavior of nonterminating systems. Most of the results on recognizable languages of finite words were extended to ω-words (see [21] for an overview). For traces, this generalization was fruitful, too [9, 6, 3]. Bloom and Ésik [1] considered the set of pomsets obtained from the singletons by the sequential and the parallel product and by the sequential ω-power. In addition, Ésik and Okawa [7] allowed the parallel ω-power. They obtained inner characterizations of the pomsets obtained this way and considered the equational theory of the corresponding algebras.

This paper deals with the set of pomsets that can be obtained by the sequential and the parallel product as well as by the infinite sequential product of pomsets. First, we show a simple characterization of these pomsets (Lemma 1). The main part of the paper is devoted to the question whether Büchi's correspondence between monadic second order logic on ω-words and recognizable sets can be transfered to the setting of (infinite) N-free pomsets. Our main result, Theorem 16, states that this is indeed possible. More precisely, we consider ω-series-rational sets, i.e. sets that can be constructed from finite sets of finite N-free pomsets by the operation of sequential and parallel concatenation, sequential iteration, sequential ω-iteration and union (without the ω-iteration, this class was considered in [15, 16]). We can show that a set of infinite N-free pomsets is ω-series-rational if and only if it can be axiomatized in monadic second order logic and is width-bounded. Our proof relies on a suitable (algebraic) definition of recognizable sets of infinite N-free pomsets and on a deep result from the theory of infinite traces [6].

Recall that Courcelle [2] considered the counting monadic second order logic on graphs of finite tree width. In this setting, a set of finite graphs is axiomatizable in Courcelle's logic if and only if it is "recognizable" [13]. It is not difficult to show that any ω-series-rational set of N-free pomsets is axiomatizable in this logic. If one tried to prove the inverse implication, i.e. started from an axiomatizable set of N-free pomsets, one would yield a rational set of terms over the parallel and the sequential product. But, as usual in term languages, this set makes use of an extended alphabet. Therefore, it is not clear how to construct a series-rational expression without additional variables from this rational term language. For this difficulty, we chose to prove our main result using traces and not Courcelle's approach.

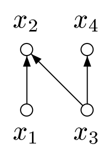
Let us finish this introduction with some open problems that call for an investigation: First, we obtained only a relation between algebraically recognizable, monadically axiomatizable, and ω-series-rational sets. It would be interesting to have a characterization in terms of branching automata, too. To this purpose, one first has to extend them in such a way that branching automata can run on infinite N-free pomsets. Second, we would have liked to incorporate the parallel iteration or even the parallel ω-power in the construction of rational sets. This easily allows the construction of sets that cannot be axiomatized in monadic second order logic. Therefore, one could try to extend the expressive power of this logic suitably.

2 Basic definitions

2.1 Order theory

Let $(V, \leq)$ be a partially ordered set. We write $x \parallel y$ for elements $x, y \in V$ if they are incomparable. A set $A \subseteq V$ is an *antichain* provided the elements of A are mutually incomparable. The *width* of the partially ordered set $(V, \leq)$ is the least cardinal $w(V, \leq)$ such that $|A| \leq w(V, \leq)$ for any antichain A. If $w(V, \leq)$ is a natural number, we say $(V, \leq)$ *has finite width.* Note that there exist partially ordered sets that contain only finite antichains, but have infinite width. We write $x \longrightarrow\!\!\!< y$ for $x, y \in V$ if $x < y$ and there is no element properly between x and y. Furthermore, $\downarrow\! y$ denotes the principal ideal $\{x \in V \mid x \leq y\}$ generated by $y \in V$.

An *N-free poset* $(V, \leq)$ is a nonempty, at most countably infinite partially ordered set such that the partially ordered set $(N, \leq_N)$ cannot be embedded into $(V, \leq)$ (cf. picture below), any antichain in $(V, \leq)$ is finite, and $\downarrow\! x$ is finite for any $x \in V$. Let Γ be an alphabet, i.e. a nonempty finite set. Then $\mathrm{NF}^\infty(\Gamma)$ denotes the set of all Γ-labeled N-free posets $(V, \leq, \lambda)$. These labeled posets are called *N-free pomsets.* Let $\mathrm{NF}(\Gamma)$ denote the set of finite N-free pomsets over Γ.



The poset $(N, \leq_N)$

Next, we define the sequential and the parallel product of Γ-labeled posets: Let $t_1 = (V_1, \leq_1, \lambda_1)$ and $t_2 = (V_2, \leq_2, \lambda_2)$ be Γ-labeled posets with $V_1 \cap V_2 = \emptyset$. The *sequential product* $t_1 \cdot t_2$ of t_1 and t_2 is the Γ-labeled partial order

$$(V_1 \cup V_2, \leq_1 \cup \leq_2 \cup V_1 \times V_2, \lambda_1 \cup \lambda_2).$$

Thus, in $t_1 \cdot t_2$, the labeled poset t_2 is put on top of the labeled poset t_1. On the contrary, the *parallel product* $t_1 \parallel t_2$ is defined to be

$$(V_1 \cup V_2, \leq_1 \cup \leq_2, \lambda_1 \cup \lambda_2),$$

i.e. here the two partial orders are set side by side. By $\mathrm{SP}(\Gamma)$, we denote the least class of Γ-labeled posets containing the singletons that is closed under the application of the sequential product $\cdot$ and the parallel product $\parallel$.

To construct infinite labeled posets, we extend the sequential product $\cdot$ naturally to an infinite one as follows: For $i \in \omega$, let $t_i = (V_i, \leq_i, \lambda_i)$ be mutually disjoint Γ-labeled posets. Then the infinite sequential product is defined by

$$\prod_{i \in \omega} t_i = (\bigcup_{i \in \omega} V_i, \bigcup_{i \in \omega} \leq_i \cup \bigcup_{\substack{i,j \in \omega \\ i<j}} V_i \times V_j, \bigcup_{i \in \omega} \lambda_i).$$

By $\mathrm{SP}^\infty(\Gamma)$, we denote the least class $\mathcal{C}$ of Γ-labeled posets such that

- $\mathrm{SP}(\Gamma) \subseteq \mathcal{C}$,
- $s, t \in \mathcal{C}$ implies $s \parallel t \in \mathcal{C}$,
- $s, t \in \mathcal{C}$ and s finite imply $s \cdot t \in \mathcal{C}$, and
- $t_i \in \mathcal{C}$ finite for $i \in \omega$ implies $\prod_{i \in \omega} t_i \in \mathcal{C}$.

Thus, a Γ-labeled poset belongs to $\mathrm{SP}^\infty(\Gamma)$ if it can be constructed from the finite Γ-labeled pomsets applying the sequential product, the parallel product or the infinite product.

Based on results from [1], we extend the known equality $\mathrm{SP}(\Gamma) = \mathrm{NF}(\Gamma)$ [10] to infinite Γ-labeled posets:

Lemma 1. *Let Γ be an alphabet. Then* $\mathrm{SP}^\infty(\Gamma) = \mathrm{NF}^\infty(\Gamma)$.

Proof. By induction on the construction of an element of $\mathrm{SP}^\infty(\Gamma)$, one shows the inclusion $\mathrm{SP}^\infty(\Gamma) \subseteq \mathrm{NF}^\infty(\Gamma)$. For the converse inclusion, let $t \in \mathrm{NF}^\infty(\Gamma)$. We may assume that t is connected. By [1, Lemma 4.10], either t is an infinite sequential product of finite N-free pomsets, or there exist $s \in \mathrm{NF}(\Gamma)$ and $t_1, t_2 \in \mathrm{NF}^\infty(\Gamma)$ with $t = s \cdot (t_1 \parallel t_2)$. Then we can proceed inductively with t_1 and t_2. This inductive decomposition will eventually terminate since antichains in t are finite. Since $\mathrm{NF}(\Gamma) = \mathrm{SP}(\Gamma)$, this finishes the proof. $\square$

The sequential, the parallel and the infinite sequential products can easily be extended to sets of (finite) N-free pomsets as follows: Let $S \subseteq \mathrm{NF}(\Gamma)$ and $S', T' \subseteq \mathrm{NF}^\infty(\Gamma)$. Then we define

$$S \cdot T' := \{s \cdot t \mid s \in S, t \in T'\}, \qquad S^+ := \{s_1 \cdot s_2 \cdots s_n \mid n > 0, s_i \in S\},$$
$$S' \parallel T' := \{s \parallel t \mid s \in S, t \in T\} \text{ and } S^\omega := \{\textstyle\prod_{i\in\omega} s_i \mid s_i \in S\}.$$

The class of *series-rational languages* [15, 16] is the least class $\mathcal{C}$ of subsets of $\mathrm{NF}(\Gamma)$ such that

- $\{s\} \in \mathcal{C}$ for $s \in \mathrm{NF}(\Gamma)$, and
- $S \cup T, S \cdot T, S \parallel T, S^+ \in \mathcal{C}$ for $S, T \in \mathcal{C}$.

Note that we do not allow the iteration of the parallel product in the construction of series-rational languages. Therefore, for any series-rational language S there exists an $n \in \omega$ with $w(s) \leq n$ for any $s \in S$, i.e. any series-rational language is *width-bounded*.

The class of *ω-series-rational languages* is the least class $\mathcal{C}$ of subsets of $\mathrm{NF}^\infty(\Gamma)$ such that

- $\{s\} \in \mathcal{C}$ for $s \in \mathrm{NF}(\Gamma)$,
- $S \cup T, S \parallel T \in \mathcal{C}$ for $S, T \in \mathcal{C}$, and
- $S^+, S^\omega, S \cdot T \in \mathcal{C}$ for $S, T \in \mathcal{C}$ and $S \subseteq \mathrm{NF}(\Gamma)$.

For the same reason as for series-rational languages, any ω-series-rational language is width-bounded. It is easily seen that the series-rational languages are precisely those ω-series-rational languages that contain only finite labeled posets.

2.2 Traces

We recall the basic definitions and some results from the theory of Mazurkiewicz traces since they will be used in our proofs, in particular in Section 4.2.

A *dependence alphabet* (Γ, D) is a finite set Γ together with a binary reflexive and symmetric relation D that is called *dependence relation*. The complementary relation $I = \Gamma^2 \setminus D$ is the *independence relation*. From a dependence alphabet, we define a binary operation $*$ on the set of Γ-labeled posets as follows: Let $t_i = (V_i, \leq_i, \lambda_i)$ be disjoint Γ-labeled posets ($i = 1, 2$). Furthermore, let $E = \{(x, y) \in V_1 \times V_2 \mid (\lambda_1(x), \lambda_2(y)) \in D\}$. Then

$$t_1 * t_2 := (V_1 \cup V_2, \leq_1 \cup (\leq_1 \circ E \circ \leq_2) \cup \leq_2, \lambda_1 \cup \lambda_2)$$

is the *trace product* of t_1 and t_2 relative to (Γ, D). Let $\mathbb{M}(\Gamma, D)$ denote the least class of Γ-labeled posets closed under the application of the trace product $*$ that contains the singletons. The elements of $\mathbb{M}(\Gamma, D)$ are called *traces*. The set $\mathbb{M}(\Gamma, D)$ together with the trace product as a binary operation is a semigroup (it is no monoid since we excluded the empty poset from our considerations). Note that $\mathbb{M}(\Gamma, D)$ consists of finite posets, only. One can show that a finite nonempty Γ-labeled poset $(V, \leq, \lambda)$

belongs to $\mathbb{M}(\Gamma, D)$ if and only if we have for any $x, y \in V$:

(a) $x \longrightarrow\!\!\!< y$ implies $(\lambda(x), \lambda(y)) \in D$, and (b) $x \parallel y$ implies $(\lambda(x), \lambda(y)) \notin D$. This characterization leads to the definition of an infinitary extension of traces: A Γ-labeled poset $(V, \leq, \lambda)$ is a *real trace* if V is at most countably infinite, $\downarrow x$ is finite for any $x \in V$, and (a) and (b) hold in $(V, \leq, \lambda)$. The set of real traces over the dependence alphabet (Γ, D) is denoted by $\mathbb{R}(\Gamma, D)$. Note that the infinite product $t_0 * t_1 * t_2 \cdots$ of finite traces $t_i \in \mathbb{M}(\Gamma, D)$ can be naturally defined and yields a real trace.

A set $L \subseteq \mathbb{R}(\Gamma, D)$ of real traces is *recognizable* [9, 8] if there exists a finite semigroup $(S, *)$ and a semigroup homomorphism $\eta : \mathbb{M}(\Gamma, D) \to (S, *)$ such that $t_0 * t_1 * t_2 \cdots \in L$ implies $s_0 * s_1 * s_2 \cdots \in L$ for any finite traces $s_i, t_i \in \mathbb{M}(\Gamma, D)$ with $\eta(t_i) = \eta(s_i)$ for $i \in \omega$.

2.3 Monadic second order logic

In this section, we will define monadic second order formulas and their interpretations over Γ-labeled pomsets. *Monadic formulas* involve first order variables $x, y, z \ldots$ for vertices and monadic second order variables $X, Y, Z, \ldots$ for sets of vertices. They are built up from the atomic formulas $\lambda(x) = a$ for $a \in \Gamma$, $x \leq y$, and $x \in X$ by means of the boolean connectives $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$ and quantifiers $\exists, \forall$ (both for first order and for second order variables). Formulas without free variables are called sentences. The satisfaction relation $\models$ between Γ-labeled posets $t = (V, \leq, \lambda)$ and monadic sentences φ is defined canonically with the understanding that first order variables range over the vertices of V and second order variables over subsets of V.

Let $\mathcal{C}$ be a set of Γ-labeled posets and φ a monadic sentence. Furthermore, let $L = \{t \in \mathcal{C} \mid t \models \varphi\}$ denote the set of posets from $\mathcal{C}$ that satisfy φ. Then we say that the sentence φ *axiomatizes the set L relative to* $\mathcal{C}$ or that L is *monadically axiomatizable relative to* $\mathcal{C}$.

In [6], it was shown that a set of real traces is recognizable if and only if it is monadically axiomatizable relative to the set of all real traces. This result generalizes Büchi's Theorem that states the same fact for ω-words.

3 From ω-series-rational to monadically axiomatizable sets

Let $t = (V, \leq, \lambda)$ be some N-free pomset and $X \subseteq V$. Since any antichain in t is finite, the set X is finite if and only if it is bounded by an antichain. Hence the formula $\exists A \forall a, b, x(((a, b \in A \wedge a \leq b) \rightarrow b \leq a) \wedge (x \in X \rightarrow \exists c : (c \in A \wedge x \leq c)))$ expresses that the set X is finite. We denote this formula, that will be useful in the following proof, by finite(X).

Lemma 2. *Let Γ be an alphabet and let $L \subseteq \mathrm{NF}^\infty(\Gamma)$ be an ω-series-rational language. Then L is monadically axiomatizable relative to $\mathrm{NF}^\infty(\Gamma)$.*

Proof. Clearly, any set $\{s\}$ with s finite can be monadically axiomatized. Now let S and T be two sets of N-free pomsets axiomatized by the monadic sentences σ and τ, respectively. Then $S \cup T$ is axiomatized by $\sigma \vee \tau$. The set $S \parallel T$ consists of all N-free pomsets satisfying

$$\exists X(X \neq \emptyset \wedge X^{co} \neq \emptyset \wedge \forall x \forall y(x \in X \wedge y \notin X \rightarrow x \parallel y) \wedge \sigma \upharpoonright X \wedge \tau \upharpoonright X^{co})$$

where $\sigma \upharpoonright X$ is the restriction of σ to the set X and $\tau \upharpoonright X^{co}$ that of τ to the complement of X. The sequential product can be dealt with similarly.

Next we show that S^+ can be described by a monadic sentence: The idea of a sentence axiomatizing S^+ is to color the vertices of an N-free pomset s by two colors such that the coloring corresponds to a factorization in factors $s = s_1 \cdot s_2 \cdot s_3 \cdots s_n$ where every factor s_i belongs to S. The identification of the S-factors will be provided by the property of being a maximal convex one-colored set. More formally, we define $\varphi = \sigma \vee \exists X \exists Y(\varphi_1 \wedge \varphi_X \wedge \varphi_Y \wedge \text{finite}(X) \wedge \text{finite}(Y))$ where φ_1 asserts that X and Y form a partition of the set of vertices such that vertices from X and vertices from Y are mutually comparable. The formula φ_X states that the maximal subsets of X that are convex satisfy σ, i.e.

$$\varphi_X = \forall Z \left[\begin{pmatrix} Z \subseteq X \wedge Z \text{ is convex} \wedge Z \neq \emptyset \wedge \\ \forall Z'(Z \subseteq Z' \subseteq X \wedge Z' \text{ is convex} \rightarrow Z = Z') \end{pmatrix} \rightarrow \sigma \restriction Z \right]$$

and the formula φ_Y is defined similarly with Y taking the place of X. Asserting that the sets X and Y are finite ensures that the sentence φ is satisfied by finite N-free pomsets, only. Hence we get indeed that φ axiomatizes S^+.

To axiomatize S^ω, we can proceed similarly to the case S^+. □

The remaining pages are devoted to the converse of the above theorem, i.e. to the question whether all monadically axiomatizable sets are ω-series-rational. Before we continue, let us sketch an idea how to tackle this problem, and explain, why we will not follow this way. Any N-free pomset is the value of a term over the signature $\{\prod, \cdot, \|\} \cup \Gamma$ (where $\prod$ has arity ω). For a set L of N-free pomsets, let T_L denote the set of all terms over the given signature whose value belongs to L. Note that T_L is a set of trees labeled by the elements from $\{\prod, \cdot, \|\} \cup \Gamma$. Similarly to [2], one can show that T_L is monadically axiomatizable whenever L is monadically axiomatizable. Hence, by the results from [20], T_L is recognizable. This implies that T_L is a rational tree language over the alphabet $\{\prod, \cdot, \|\} \cup \Gamma \cup X$ for some finite set X of additional symbols. Since I do not know how to infer that L is series-rational in case T_L is rational over an extended alphabet, I follow another way in the proof of the converse of the above theorem.

4 From monadically axiomatizable to ω-recognizable sets

4.1 ω-recognizable sets

Recall that a set of infinite words $L \subseteq \Gamma^\omega$ is Büchi-recognizable if there exists a finite semigroup $(S, *)$ and a semigroup homomorphism $\eta : \Gamma^+ \to (S, *)$ such that for any $u_i, v_i \in \Gamma^*$ with $\eta(u_i) = \eta(v_i)$ for $i \in \omega$, we have $u_0 u_1 u_2 \cdots \in L$ if and only if $v_0 v_1 v_2 \cdots \in L$ (cf. [17]). Here, we use this characterization as a definition and transfer it into the context of N-free pomsets:

Let S be a set that is equipped with two binary operations $\cdot$ and $\|$. We assume these two operations to be associative and, in addition, $\|$ to be commutative. Then $(S, \cdot, \|)$ is an *sp-algebra*. Note that the set of finite N-free pomsets is an sp-algebra. Mappings between sp-algebras that commute with the two products will be called *sp-homomorphisms*.

Let X be a set of variables that will range over elements of $\text{NF}(\Gamma)$. We call the terms over $\cdot$ and $\|$ that contain variables in X *finite terms*. Now let t_i be finite terms for $i \in \omega$. Then $\prod_{i \in \omega} t_i$ is a *term* and any finite term is a term. Furthermore, if t is a finite term and t_i are terms for $1 \leq i \leq n$, then $t \cdot t_1$ and $t_1 \| t_2 \| \cdots t_n$ are terms, too. Now let $f : X \to \text{NF}(\Gamma)$. Then $f(t) \in \text{NF}^\infty(\Gamma)$ is defined naturally for any term t. Let $L \subseteq \text{NF}^\infty(\Gamma)$. Then L is *ω-recognizable* if there exists a finite sp-algebra $(S, \cdot, \|)$ and an sp-homomorphism $\eta : \text{NF}(\Gamma) \to (S, \cdot, \|)$ such that for any term t and any mappings $f, g : X \to \text{NF}(\Gamma)$ with $\eta \circ f = \eta \circ g$, we have $f(t) \in L$ if and only if $g(t) \in L$. In this case, we will say that the sp-homomorphism η *recognizes* L. In [15], recognizable subsets of $\text{NF}(\Gamma)$ are defined: A set $L \subseteq \text{NF}(\Gamma)$ is *recognizable* if there exists a finite sp-algebra $(S, \cdot, \|)$ and an sp-homomorphism $\eta : \mathrm{NF}(\Gamma) \to (S, \cdot, \|)$ such that $L = \eta^{-1}\eta(L)$. One can easily check that $L \subseteq \mathrm{NF}(\Gamma)$ is recognizable in this sense if and only if it is ω-recognizable.

Example 3. Let $(V, \leq)$ be a tree without maximal elements, such that $\downarrow v$ is finite for any $v \in V$, any node has at most 2 upper neighbors, and almost all nodes from V have only one upper neighbor. Let n be the number of branching points of $(V, \leq)$. Then we call $(V, \leq)$ a *tree with n branching points*. Note that $(V, \leq, \lambda)$ is an N-free pomset.

Now let N be a set of natural numbers and let L_N denote the set of all Γ-labeled trees with n branching points for some $n \in N$. We show that L_N is ω-recognizable:

We consider the sp-algebra $S = \{1, 2\}$ with the mapping $\eta : \mathrm{NF}(\Gamma) \to S$ defined by $\eta(t) = \min(w(t), 2)$ for any $t \in \mathrm{NF}(\Gamma)$. To obtain an sp-homomorphism, let $x \| y = \min(2, x + y)$ and $x \cdot y = \max(x, y)$ for any $x, y \in S$. Now let T be a term and $f, g : X \to \mathrm{NF}(\Gamma)$ with $\eta \circ f = \eta \circ g$. Furthermore, assume $f(T) \in L_N$, i.e. that $f(T)$ is a tree with $n \in N$ branching points. As $f(T)$ has no leaves, every parallel product $\|$ in T is applied to two non-finite terms and similarly the second factor of every sequential product $\cdot$ in T is a non-finite term. Hence every variable x_i (that occurs in T at all) occurs in T either as a left factor of a sequential product $\cdot$ or within the scope of an infinite product $\|$. Since $f(T)$ is a tree, this implies that $f(x_i)$ is a (finite) linear order, i.e. $w(f(x_i)) = 1$. Now $\eta \circ f = \eta \circ g$ implies $w(g(x_i)) = 1$. Hence the N-free pomset $g(T)$ differs from the tree with n branching points $f(T)$ only in some non-branching pieces. Thus, $g(T)$ is a tree with n branching points, i.e. $g(T) \in L_N$ as required. Hence we showed that L_N is indeed ω-recognizable.

By the example above, the number of ω-recognizable subsets of $\mathrm{NF}^\infty(\Gamma)$ is $2^{\aleph_0}$. Since there are only countably many monadic sentences or ω-series-rational languages, not all ω-recognizable sets are monadically axiomatizable or ω-series-rational. Later, we will see that these three notions coincide for width-bounded sets. But first, we show that any ω-recognizable set of N-free pomsets of finite width is of a special form (cf. Proposition 6).

Let $(S, \cdot, \|)$ be an sp-algebra. Then a pair $(s, e) \in S^2$ is *linked* if $s \cdot e = s$ and $e \cdot e = e$. A *simple term* of order 1 is an element of S or a linked pair (s, e). Now let $n > 1$, σ_i for $i = 1, 2, \ldots n$ be simple terms of order n_i, and $s \in S$. Then $s \cdot (\sigma_1 \| \sigma_2 \| \cdots \sigma_n)$ is a simple term of order $n_1 + n_2 + \ldots n_n$.

For an sp-homomorphism $\eta : \mathrm{NF}(\Gamma) \to S$ and a simple term σ, we define the language $L_\eta(\sigma)$ inductively: If $\sigma \in S$, we set $L_\eta(\sigma) := \eta^{-1}(\sigma)$. For a linked pair (s, e), we define $L_\eta(s, e) := \eta^{-1}(s) \cdot (\eta^{-1}(e))^\omega$. Furthermore, $L_\eta(s \cdot (\sigma_1 \| \sigma_2 \| \ldots \sigma_n)) := \eta^{-1}(s) \cdot (L_\eta(\sigma_1) \| L_\eta(\sigma_2) \| \cdots L_\eta(\sigma_n))$.

Lemma 4. *Let Γ be an alphabet, $(S, \cdot, \|)$ a finite sp-algebra and $\eta : \mathrm{NF}(\Gamma) \to (S, \cdot, \|)$ an sp-homomorphism. Let furthermore $t \in \mathrm{NF}^\infty(\Gamma)$ be an N-free pomset of finite width. Then there exist simple terms $\tau_1, \tau_2, \ldots, \tau_m$ of order at most $w(t)$ with $n \leq w(t)$ and $t \in L_\eta(\tau_1) \| L_\eta(\tau_2) \| \cdots L_\eta(\tau_m)$.*

Proof. If t is finite, the lemma is obvious since the element $s = \eta(t)$ of S is a simple term of order 1. Thus, we may assume t to be infinite. First consider the case that $t = \prod_{i \in \omega} t_i$ is an infinite product of finite N-free pomsets t_i. Let $s_i := \eta(t_i)$. A standard application of Ramsey's Theorem [19] (cf. also [17]) yields the existence of positive integers n_i for $i \in \omega$ and a linked pair $(s, e) \in S^2$ such that $s = s_0 s_1 \cdots s_{n_0}$ and $e = s_{n_i+1} \cdot s_{n_i+2} \cdots s_{n_{i+1}}$ for $i \in \omega$. Hence $t \in L_\eta(s, e)$. Since (s, e) is a simple term of order $1 \leq w(t)$, we showed the lemma for infinite products of finite N-free pomsets.

Now the proof proceeds by induction on the width $w(t)$ of an N-free pomset t. By [1], t is either a parallel product, or an infinite sequential product, or of the form $s \cdot (t_1 \parallel t_2)$ for some finite N-free pomset s. In any of these cases, one uses the induction hypothesis (which is possible since e.g. $w(t_1) < w(t)$ in the third case). $\square$

The following lemma shows that the set $L_\eta(\tau)$ is contained in any ω-recognizable set L that intersects $L_\eta(\tau)$.

Lemma 5. *Let Γ be an alphabet, $(S, \cdot, \parallel)$ a finite sp-algebra, and $\eta : \mathrm{NF}(\Gamma) \to (S, \cdot, \parallel)$ an sp-homomorphism and τ_i a simple term for $1 \le i \le m$. Let furthermore $t, t' \in L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_m)$. Then there exist a term T and mappings $f, g : X \to \mathrm{NF}(\Gamma)$ with $\eta \circ f = \eta \circ g$ such that $f(T) = t$ and $g(T) = t'$.*

Proof. First, we show the lemma for the case $m = 1$ (for simplicity, we write τ for τ_1): In this restricted case, the lemma is shown by induction on the construction of the simple term τ. For $\tau = s \in S$, the term $T = x$ and the mappings $f(y) = t$ and $g(y) = t'$ for any $y \in X$ have the desired properties. For a linked pair $\tau = (s, e)$, the term $T = \prod_{i\in\omega} x_i$ can be used to show the statement. For $\tau = s \cdot (\tau_1 \parallel \tau_2 \parallel \cdots \tau_n)$, one sets $T = x(T_1 \parallel T_2 \parallel \cdots T_n)$ where $x \in X$ and T_i is a term that corresponds to τ_i. We can assume that no variable occurs in T_i and in T_j for $i \neq j$ and that x does not occur in any of the terms T_i. Then the functions f_i and g_i, that exist by the induction hypothesis, can be joint which results in functions f and g satisfying the statement of the lemma. $\square$

Let L be an ω-recognizable set of N-free pomsets of finite width. The following proposition states that L is the union of languages of the form $L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_m)$. But this union might be infinite. The proof is immediate by Lemmas 4 and 5.

Proposition 6. *Let Γ be an alphabet and $L \subseteq \mathrm{NF}^\infty(\Gamma)$ be a set of N-free pomsets of finite width. Let L be recognized by the sp-homomorphism $\eta : \mathrm{NF}(\Gamma) \to (S, \cdot, \parallel)$, and let $\mathfrak{T}$ denote the set of finite tuples of simple terms $(\tau_1, \tau_2, \ldots, \tau_m)$ such that $\emptyset \neq L \cap (L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_m))$. Then*

$$L = \bigcup_{(\tau_1,\tau_2,\ldots,\tau_m)\in\mathfrak{T}} (L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_m)).$$

4.2 Monadically axiomatizable sets of bounded width

Now, we concentrate on sets of N-free pomsets whose width is *uniformly* bounded by some integer. It is our aim to show that a set of bounded width that is monadically axiomatizable relative to $\mathrm{NF}^\infty(\Gamma)$ is ω-recognizable.

Let $(V, \le)$ be a partially ordered set. Following [12], we define a directed graph $(V, E) = \mathrm{spine}(V, \le)$, called *spine*, of $(V, \le)$ as follows: The edge relation is a subset of the strict order $<$. A pair (x, y) with $x < y$ belongs to E if either $x \longrightarrow\!\!\!< y$ or for any $z \in V$ we have $x \longrightarrow\!\!\!< z \Rightarrow z \le y$ as well as $z \longrightarrow\!\!\!< y \Rightarrow x \le z$. Thus, $(x, y) \in E$ if either $x \longrightarrow\!\!\!< y$ or $x < y$ and any upper neighbor of x (any lower neighbor of y) is below y (above x, respectively).

Let $\mathrm{maxco}(\mathrm{spine}(V, \le))$ denote the maximal size of a totally unconnected set of vertices in the graph $\mathrm{spine}(V, \le)$. The restriction of the following lemma to finite partially ordered sets was shown in [12]. The extension we use here is an obvious variant of this result:

Lemma 7 (cf. [12]). *Let $n > 0$. There exists a dependence alphabet (Γ_n, D_n) with the following property: Let $(V, \le)$ be a poset with $\mathrm{maxco}(\mathrm{spine}(V, \le)) \le n$ such that ${\downarrow}x$ is finite for any $x \in V$. Then there exists a mapping $\lambda : V \to \Gamma_n$ such that $(V, \le, \lambda) \in \mathbb{R}(\Gamma_n, D_n)$ is a real trace over the dependence alphabet (Γ_n, D_n).*

We introduced the spine of a partially ordered set and mentioned the result of Hoogeboom and Rozenberg since the spine of an N-free partially ordered set is "small" as the following lemma states.

Lemma 8. *Let Σ be an alphabet and $t = (V, \leq, \lambda) \in \mathrm{NF}^\infty(\Sigma)$ be an N-free pomset of finite width. Then* $\mathrm{maxco}(\mathrm{spine}(t)) < 2w(t) \cdot (w(t) + 1)$.

Let Γ and Σ be two alphabets. A set of Σ-labeled posets L is the *projection* of a set M of Γ-labeled posets if there exists a mapping $\pi : \Gamma \to \Sigma$ such that $L = \{(V, \leq, \pi \circ \lambda) \mid (V, \leq, \lambda) \in M\}$, i.e. L is the set of relabeled (w.r.t. π) posets from M.

Now let L be a set of N-free pomsets of width at most n. Then the two lemmas above show that L is the projection of a set of real traces over a finite dependence alphabet. Because of this relation, we now start to consider sets of N-free real traces:

Languages of N-free real traces. Recall that we want to show that any monadically axiomatizable set of N-free pomsets is ω-recognizable. By [6], any monadically axiomatizable set of (N-free) real traces is recognizable. Therefore, we essentially have to show that any set $L \subseteq \mathbb{R}(\Gamma, D)$ of N-free real traces that is recognizable in $\mathbb{R}(\Gamma, D)$, is ω-recognizable in $\mathrm{NF}(\Gamma)$. Thus, in particular we have to construct from a semigroup homomorphism $\eta : \mathbb{M}(\Gamma, D) \to (S, *)$ into a finite semigroup $(S, *)$ an sp-homomorphism $\gamma : \mathrm{NF}(\Gamma) \to (S^+, \cdot, \|)$ into some finite sp-algebra. This is the content of Lemma 10 that is prepared by the following definition and lemma.

Let $\mathrm{alph}^3(V, \leq, \lambda) := (\lambda \circ \min(V, \leq), \lambda(V), \lambda \circ \max(V, \leq))$ for any finite Γ-labeled poset $(V, \leq, \lambda)$. Let $\mathcal{C}$ be a set of finite Γ-labeled posets (the two examples, we will actually consider, are $\mathcal{C} = \mathrm{NF}(\Gamma)$ and $\mathcal{C} = \mathbb{M}(\Gamma, D)$) and $\eta : \mathcal{C} \to S$ a mapping. Then η is *strongly alphabetic* if it is surjective and if $\eta(t_1) = \eta(t_2)$ implies $\mathrm{alph}^3(t_1) = \mathrm{alph}^3(t_2)$ for any $t_i = (V_i, \leq, \lambda_i) \in \mathcal{C}$. Using the concept of a dependence chain, one can easily show the existence of a semigroup homomorphism η into a finite semigroup such that η is strongly alphabetic:

Lemma 9. *Let (Γ, D) be a dependence alphabet. There exists a finite semigroup $(S, *)$ and a* strongly alphabetic *semigroup homomorphism $\eta : \mathbb{M}(\Gamma, D) \to (S, *)$.*

Lemma 10. *Let (Γ, D) be a dependence alphabet and $\eta : \mathbb{M}(\Gamma, D) \to (S, *)$ be a strongly alphabetic semigroup homomorphism into a finite semigroup $(S, *)$. Then there exists a finite sp-algebra $(S^+, \cdot, \|)$ with $S \subset S^+$ and a strongly alphabetic sp-homomorphism $\gamma : \mathrm{NF}(\Gamma) \to (S^+, \cdot, \|)$ such that*

1. *$\eta(t) = \gamma(t)$ for $t \in \mathbb{M}(\Gamma, D) \cap \mathrm{NF}(\Gamma)$ and*
2. *$\gamma(t) \in S$ implies $t \in \mathbb{M}(\Gamma, D) \cap \mathrm{NF}(\Gamma)$ for any $t \in \mathrm{NF}(\Gamma)$.*

Proof. Since η is strongly alphabetic, there is a function $\mathrm{alph}^3 : S \to (\mathcal{P}(\Gamma) \setminus \{\emptyset\})^3$ with $\mathrm{alph}^3(\eta(t)) = \mathrm{alph}^3(t)$ for any trace $t \in \mathbb{M}(\Gamma, D)$. From the semigroup $(S, *)$ and the function alph^3, we define an sp-algebra $(S_1, \cdot, \|)$ as follows: Let $S_1 = S \dot\cup (\mathcal{P}(\Gamma) \setminus \{\emptyset\})^3$ and extend the function alph^3 to S_1 by $\mathrm{alph}^3(X) = X$ for $X \in (\mathcal{P}(\Gamma) \setminus \{\emptyset\})^3$. Now let

$$x \cdot y = \begin{cases} x * y & \text{if } \pi_3 \circ \mathrm{alph}^3(x) \times \pi_1 \circ \mathrm{alph}^3(y) \subseteq D \\ (\pi_1 \circ \mathrm{alph}^3(x), \pi_2 \circ \mathrm{alph}^3(x) \cup \pi_2 \circ \mathrm{alph}^3(y), \pi_3 \circ \mathrm{alph}^3(y)) & \text{otherwise} \end{cases}$$

$$x \parallel y = \begin{cases} x * y & \text{if } \pi_2 \circ \mathrm{alph}^3(x) \times \pi_2 \circ \mathrm{alph}^3(y) \subseteq I \\ \mathrm{alph}^3(x) \cup^3 \mathrm{alph}^3(y) & \text{otherwise} \end{cases}$$

for any $x, y \in S$ where $\cup^3$ is the componentwise union of elements of $(\mathcal{P}(\Gamma) \setminus \{\emptyset\})^3$. Let furthermore $x \cdot X = X \cdot x = x \parallel X = X \parallel x = X \cup^3 \mathrm{alph}^3(x)$ for any $x \in S_1$ and

$X \in (\mathcal{P}(\Gamma) \setminus \{\emptyset\})^3$. One can easily check that the mappings $\cdot$ and $\|$ are associative and that the parallel product $\|$ is commutative. Now let $\gamma : \mathrm{NF}(\Gamma) \to S_1$ be defined by $\gamma(t) = \eta(t)$ for $t \in \mathbb{M}(\Gamma, D) \cap \mathrm{NF}(\Gamma)$ and $\gamma(t) = \mathrm{alph}^3(t)$ for $t \in \mathrm{NF}(\Gamma) \setminus \mathbb{M}(\Gamma, D)$ and let S^+ be the image of γ. Then γ is a strongly alphabetic sp-homomorphism onto $(S^+, \cdot, \|)$. ☐

Lemma 11. *Let (Γ, D) be a dependence alphabet. Then $\mathbb{R}(\Gamma, D) \cap \mathrm{NF}^\infty(\Gamma)$ is ω-recognizable.*

Proof. By Lemma 9, there exists a strongly alphabetic semigroup homomorphism $\alpha : \mathbb{M}(\Gamma, D) \to (T, *)$ into a finite semigroup $(T, *)$. Then, by Lemma 10, we find a strongly alphabetic sp-homomorphism $\eta : \mathrm{NF}(\Gamma) \to (S, \cdot, \|)$ that coincides with α on $\mathbb{M}(\Gamma, D) \cap \mathrm{NF}(\Gamma)$ such that $\eta(t) \in T$ implies $t \in \mathbb{M}(\Gamma, D) \cap \mathrm{NF}(\Gamma)$. Let furthermore $f, g : X \to \mathrm{NF}(\Gamma)$ be functions with $\eta \circ f = \eta \circ g$. By induction on the construction of a term t, one shows that $f(t)$ is a trace if and only $g(t)$ is a trace. ☐

From a term t, we construct a finite or infinite sequence $\mathrm{lin}(t)$ over X inductively: First, $\mathrm{lin}(x_i) = (x_i)$. Now let t_i for $i \in \omega$ be finite terms. Then $\mathrm{lin}(t_1 \cdot t_2) = \mathrm{lin}(t_1 \parallel t_2)$ is the concatenation of the sequences $\mathrm{lin}(t_1)$ and $\mathrm{lin}(t_2)$. Similarly, $\mathrm{lin}(\prod_{i \in \omega} t_i)$ is the concatenation of the sequences $\mathrm{lin}(t_i)$. Now let t_1, t_2 be terms and t be a finite term. Then $\mathrm{lin}(t \cdot t_1)$ is the concatenation of the sequences $\mathrm{lin}(t)$ and $\mathrm{lin}(t_1)$ (note that $\mathrm{lin}(t)$ is finite). If $\mathrm{lin}(t_1)$ is finite, let $\mathrm{lin}(t_1 \parallel t_2)$ be the concatenation of $\mathrm{lin}(t_1)$ with $\mathrm{lin}(t_2)$ and, if $\mathrm{lin}(t_1)$ is infinite but $\mathrm{lin}(t_2)$ is finite, let $\mathrm{lin}(t_1 \parallel t_2)$ be the concatenation of $\mathrm{lin}(t_2)$ with $\mathrm{lin}(t_1)$. If both $\mathrm{lin}(t_1)$ and $\mathrm{lin}(t_2)$ are infinite, $\mathrm{lin}(t_1 \parallel t_2)$ is the alternating sequence $(y_1^1, y_1^2, y_2^1, y_2^2, \dots)$ with $\mathrm{lin}(t_i) = (y_1^i, y_2^i, \dots)$ for $i = 1, 2$.

For a term t with $\mathrm{lin}(t) = (y_1, y_2, y_3, \dots)$ and a mapping $f : X \to \mathrm{NF}(\Gamma)$, let $\bigstar(f, t) := f(y_1) * f(y_2) * f(y_3) \cdots$ denote the infinite trace product of the pomsets $f(y_i)$. Note that $\bigstar(f, t)$ is a Γ-labeled poset that in general need not be a trace nor an N-free pomset. The following lemma implies that in certain cases $\bigstar(f, t)$ is a real trace. It is shown by induction on the depth of a term.

Lemma 12. *Let (Γ, D) be a dependence alphabet. Let t be a term and $f : X \to \mathrm{NF}(\Gamma)$. If $f(t)$ is a real trace, then $f(t) = \bigstar(f, t)$.*

Now we can show that at least any monadically axiomatizable set of N-free real traces is ω-recognizable:

Proposition 13. *Let (Γ, D) be a dependence alphabet and $\varkappa$ be a monadic sentence. Then the set $L = \{t \in \mathbb{R}(\Gamma, D) \cap \mathrm{NF}^\infty(\Gamma) \mid t \models \varkappa\}$ is ω-recognizable*

Proof. By [6], the set L is a recognizable language of real traces. Hence there is a semigroup homomorphism $\eta : \mathbb{M}(\Gamma, D) \to (S, *)$ into a finite semigroup $(S, *)$ such that, for any sequences x_i, y_i of finite traces with $\eta(x_i) = \eta(y_i)$, we have $x_1 * x_2 * \cdots \in L$ if and only if $y_1 * y_2 * \cdots \in L$. By Lemma 9, we may assume that η is strongly alphabetic. By Lemma 10, there exists a finite sp-algebra $(S^+, \cdot, \|)$ and a strongly alphabetic sp-homomorphism $\eta^+ : \mathrm{NF}(\Gamma) \to S^+$ that coincides with η on $\mathbb{M}(\Gamma, D) \cap \mathrm{NF}(\Gamma)$.

By Lemma 11, there exists an sp-homomorphism $\delta : \mathrm{NF}(\Gamma, D) \to (T, \cdot, \|)$ that recognizes $\mathbb{R}(\Gamma, D) \cap \mathrm{NF}(\Gamma)$. Now let $\alpha = \eta^+ \times \delta : \mathrm{NF}(\Gamma) \to S \times T$. We show that α recognizes L:

Let t be a term with $\mathrm{lin}(t) = (y_1, y_2, \dots)$ and $f, g : X \to \mathrm{NF}(\Gamma)$ be mappings with $\alpha \circ f = \alpha \circ g$. Suppose $f(t) \in L$. Then $f(t)$ is a real trace. Since $\delta \circ f = \delta \circ g$ and δ recognizes $\mathbb{R}(\Gamma, D) \cap \mathrm{NF}^\infty(\Gamma)$, the Γ-labeled poset $g(t)$ is a real trace, too.

From Lemma 12, we obtain $f(t) = \bigstar(f,t)$ and $g(t) = \bigstar(g,t)$. Since $f(t) \in L$, we have in particular $f(y_1) * f(y_2) * f(y_3) \cdots \in L$. Note that $f(y_i), g(y_i) \in \mathbb{M}(\Gamma, D) \cap \mathrm{NF}(\Gamma)$ and that $\eta^+ \circ f = \eta^+ \circ g$. Since η^+ and η coincide on $\mathbb{M}(\Gamma, D) \cap \mathrm{NF}(\Gamma)$, this implies $\eta(f(y_i)) = \eta(g(y_i))$. Since η recognizes the language of real traces L, we obtain $g(y_1) * g(y_2) * g(y_3) \cdots \in L$. ◻

Languages of N-free pomsets. Following Lemma 8, we explained that any width-bounded set of N-free pomsets is the projection of a set of N-free real traces over some dependence alphabet. This is the crucial point in the following proof.

Proposition 14. *Let $m \in \omega$, let Σ be an alphabet and φ a monadic sentence over Σ. Then the set $L = \{t \in \mathrm{NF}^\infty(\Sigma) \mid t \models \varphi$ and $w(t) \leq m\}$ is ω-recognizable.*

Proof. Let (Γ_n, D_n) be the dependence alphabet from Lemma 7 with $n = 2m(m+1)$. Now let $\Gamma = \Gamma_n \times \Sigma$ and $((A,a),(B,b)) \in D$ iff $(A,B) \in D_n$ for any $(A,a),(B,b) \in \Gamma$. Let $\Pi(V, \leq, \lambda) = (V, \leq, \pi_2 \circ \lambda)$ be the canonical projection from Γ-labeled posets to the set of Σ-labeled posets and consider the set $K := \{t \in \mathbb{R}(\Gamma, D) \mid \Pi(t) \in L\}$.

By Lemma 8, $\Pi(K) = L$. In the monadic sentence φ, replace any subformula of the form $\lambda(x) = a$ by $\bigvee_{A \in \Gamma_n} \lambda(x) = (A, a)$ and denote the resulting sentence by $\varkappa$. Note that $\varkappa$ is a monadic sentence over the alphabet Γ and that $K = \{t \in \mathbb{M}(\Gamma, D) \mid t \models \varkappa\}$. Since $K \subseteq \mathrm{NF}^\infty(\Gamma)$, we can apply Proposition 13, and obtain that K is ω-recognizable. Now one can show that the class of ω-recognizable languages is closed under projections which gives the desired result. ◻

5 From ω-recognizable to ω-series-rational sets

To finish the proof of our main theorem, it remains to show that any ω-recognizable set whose elements have a uniformly bounded width is ω-series-rational. This result is provided by the following proposition:

Proposition 15. *Let Γ be an alphabet and $L \subseteq \mathrm{NF}^\infty(\Gamma)$ be ω-recognizable and width-bounded. Then L is ω-series-rational.*

Proof. Let $\eta : \mathrm{NF}(\Gamma) \to (S, \cdot, \|)$ be an sp-homomorphism that recognizes L. Furthermore, let $n \in \omega$ such that $w(t) \leq n$ for any $t \in L$. Now let $\mathfrak{T}$ denote the set of $(\leq n)$-tuples of simple terms $(\tau_1, \tau_2, \ldots, \tau_k)$ of order at most n such that $\emptyset \neq L \cap (L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_k))$. Note that $\mathfrak{T}$ is finite. The proof of Proposition 6 yields that L is the union of the languages $L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_k)$ over all tuples from $\mathfrak{T}$.

Hence it remains to show that $L_\eta(\tau)$ is ω-series-rational for any simple term τ such that there exists $(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathfrak{T}$ with $\sigma_i = \tau$ for some $1 \leq i \leq k$. This proof proceeds by induction on the subterms of τ and uses in particular the fact that any width-bounded and recognizable set in $\mathrm{NF}(\Gamma)$ is series-rational [15]. ◻

Now our main theorem follows from Lemma 2, Propositions 14 and 15.

Theorem 16. *Let Γ be an alphabet and $L \subseteq \mathrm{NF}^\infty(\Gamma)$. Then the following are equivalent:*

1. *L is ω-series-rational.*
2. *L is monadically axiomatizable relative to $\mathrm{NF}^\infty(\Gamma)$ and width-bounded.*
3. *L is ω-recognizable and width-bounded.*

Recall that a set of finite N-free pomsets is recognizable in the sense of [15] if and only if it is ω-recognizable. Therefore, a direct consequence of the main theorem (together with the results from [15, 16] where the remaining definitions can be found) is the following

Corollary 17. *Let Γ be an alphabet and $L \subseteq \mathrm{NF}(\Gamma)$ be width-bounded. Then L can be accepted by a branching automaton iff it is recognizable iff series-rational iff monadically axiomatizable.*

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