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The Isomorphism Problem for ω-Automatic Trees

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Abstract

The main result of this paper states that the isomorphism problem for ω-automatic trees of finite height is at least has hard as second-order arithmetic and therefore not analytical. This strengthens a recent result by Hjorth, Khoussainov, Montalbán, and Nies [12] showing that the isomorphism problem for ω-automatic structures is not in Σ^1_2. Moreover, assuming the continuum hypothesis **CH**, we can show that the isomorphism problem for ω-automatic trees of finite height is recursively equivalent with second-order arithmetic. On the way to our main results, we show lower and upper bounds for the isomorphism problem for ω-automatic trees of every finite height: (i) It is decidable (Π^0_1-complete, resp.,) for height 1 (2, resp.), (ii) Π^1_1-hard and in Π^1_2 for height 3, and (iii) Π^1_{n-3}- and Σ^1_{n-3}-hard and in Π^1_{2n-4} (assuming **CH**) for height $n \geq 4$. All proofs are elementary and do not rely on theorems from set theory.

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1. Introduction

A graph is computable if its domain is a computable set of natural numbers and the edge relation is computable as well. Hence, one can compute effectively in the graph. On the other hand, practically all other properties are undecidable for computable graphs (e.g., reachability, connectedness, and even the existence of isolated nodes). In particular, the isomorphism problem is highly undecidable in the sense that it is complete for Σ^1_1 (the first existential level of the analytical hierarchy [25]); see e.g. [5, 10] for further investigations of the isomorphism problem for computable structures. These algorithmic deficiencies have motivated in computer science the study of more restricted classes of finitely presented infinite graphs. For instance, pushdown graphs, equational graphs, and prefix recognizable graphs have a decidable monadic second-order theory and for the former two the isomorphism problem is known to be decidable [7] (for prefix recognizable graphs the status of the isomorphism problem seems to be open).

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Automatic graphs [16] are in between prefix recognizable and computable graphs. In essence, a graph is automatic if the elements of the universe can be represented as strings from a regular language and the edge relation can be recognized by a finite state automaton with several heads that proceed synchronously. Automatic graphs (and more general, automatic structures) received increasing interest over the last years [3, 13, 17, 18, 29, 1]. One of the main motivations for investigating automatic graphs is that their first-order theories can be decided uniformly (i.e., the input is an automatic presentation and a first-order sentence). On the other hand, the isomorphism problem for automatic graphs is Σ^1_1-complete [17] and hence as complex as for computable graphs (see [23] for the recursion theoretic complexity of other natural properties of automatic graphs).

In our recent paper [21], we studied the isomorphism problem for restricted classes of automatic graphs. Among other results, we proved that: (i) the isomorphism problem for automatic trees of height at most $n \geq 2$ is complete for the level Π^0_{2n-3} of the arithmetical hierarchy, (ii) that the isomorphism problem for well-founded automatic order trees is recursively equivalent to true arithmetic, and (iii) that the isomorphism problem for automatic order trees is Σ^1_1-complete. In this paper, we extend our techniques from [21] to *ω-automatic trees*. The class of ω-automatic structures was introduced in [2]; it generalizes automatic structures by replacing ordinary finite automata by Büchi automata on ω-words. In this way, uncountable graphs can be specified. Some recent results on ω-automatic structures can be found in [22, 12, 14, 19]. On the logical side, many of the positive results for automatic structures carry over to ω-automatic structures [2, 14]. On the other hand, the isomorphism problem of ω-automatic structures is more complicated than that of automatic structures (which is Σ^1_1-complete). Hjorth et al. [12] constructed two ω-automatic structures for which the existence of an isomorphism depends on the axioms of set theory. Using Schoenfield's absoluteness theorem, they infer that isomorphism of ω-automatic structures does not belong to Σ^1_2. The extension of our elementary techniques from [21] to ω-automatic trees allows us to show directly (without a "detour" through set theory) that the isomorphism problem for ω-automatic trees of finite height is not analytical (i.e., does not belong to any of the levels Σ^1_n). For this, we prove that the isomorphism problem for ω-automatic trees of height $n \geq 4$ is hard for both levels Σ^1_{n-3} and Π^1_{n-3} of the analytical hierarchy (our proof is uniform in n). A more precise analysis moreover reveals at which height the complexity jump for ω-automatic trees occurs: For automatic as well as for ω-automatic trees of height 2, the isomorphism problem is Π^0_1-complete and hence arithmetical. But the isomorphism problem for ω-automatic trees of height 3 is hard for Π^1_1 (and therefore outside of the arithmetical hierarchy) while the isomorphism problem for automatic trees of height 3 is Π^0_3-complete [21]. Our lower bounds for ω-automatic trees even hold for the restricted class of injectively ω-automatic trees.

We prove our results by reductions from monadic second-order (fragments of) number theory. The first step in the proof is a normal form for analytical predicates. The basic idea of the reduction then is that a subset $X \subseteq \mathbb{N}$ can be encoded by an ω-word w_X over $\{0, 1\}$, where the i-th symbol is 1 if and only if $i \in X$. The combination of this basic observation with our techniques from [21] allows us to encode monadic second-order formulas over $(\mathbb{N}, +, \times)$ by ω-automatic trees of finite height. This yields the lower bounds mentioned above. We also give an upper bound for the isomorphism problem: for ω-automatic trees of height n, the isomorphism problem belongs to Π^1_{2n-4}. While the lower bound holds in the usual system **ZFC** of set theory, we can prove the upper bound only assuming in addition the continuum hypothesis. The precise recursion theoretic complexity of the isomorphism problem for ω-automatic trees remains open, it might depend on the underlying axioms for set theory.

Related work. Results on isomorphism problems for various subclasses of automatic structures can be found in [17, 18, 21, 28]. Some completeness results for low levels of the analytical hierarchy for decision problems on infinitary rational relations were shown in [8]. In [9], it was shown that the isomorphism problems for ω-tree-automatic boolean algebras, (commutative) rings, and nilpotent groups of class $n > 1$ neither belong to Σ^1_2 nor to Π^1_2.

2. Preliminaries

Let $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$ be the set of naturals without 0. With $\overline{x}$ we denote a tuple $(x_1, \ldots, x_m)$ of variables, whose length m does not matter.

2.1. The analytical hierarchy

In this paper we follow the definitions of the arithmetical and analytical hierarchy from [25]. In order to avoid some technical complications, it is useful to exclude 0 in the following, i.e., to consider subsets of $\mathbb{N}_+$. In the following, f_i ranges over unary functions on $\mathbb{N}_+$, X_i over subsets of $\mathbb{N}_+$, and $u, x, y, z, x_i, \ldots$ over elements of $\mathbb{N}_+$. The class $\Sigma^0_n \subseteq 2^{\mathbb{N}_+}$ is the collection of all sets $A \subseteq \mathbb{N}_+$ of the form

$$A = \{x \in \mathbb{N}_+ \mid (\mathbb{N}, +, \times) \models \exists y_1 \; \forall y_2 \cdots Q y^1_n, y^2_n, \ldots, y^m_n : \varphi(x, y_1, \ldots, y^1_n, y^2_n, \ldots, y^m_n)\},$$

where $Q = \forall$ (resp. $Q = \exists$) if n is even (resp. odd) and φ is a quantifier-free formula over the signature containing $+$ and $\times$. The class Π^0_n is the class of all complements of Σ^0_n sets. The classes Σ^0_n, Π^0_n ($n \geq 1$) make up the *arithmetical hierarchy*.

The analytical hierarchy extends the arithmetical hierarchy and is defined analogously using function quantifiers: The class $\Sigma^1_n \subseteq 2^{\mathbb{N}_+}$ is the collection of all sets $A \subseteq \mathbb{N}_+$ of the form

$$A = \{x \in \mathbb{N}_+ \mid (\mathbb{N}, +, \times) \models \exists f_1 \; \forall f_2 \cdots Q f_n : \varphi(x, f_1, \ldots, f_n)\}, \tag{1}$$

where $Q = \forall$ (resp. $Q = \exists$) if n is even (resp. odd) and φ is a first-order formula over the signature containing $+$, $\times$, and the functions $f_1, \ldots, f_n$. The class Π^1_n is the class of all complements of Σ^1_n sets. The classes Σ^1_n, Π^1_n ($n \geq 1$) make up the *analytical hierarchy*, see Figure 1 for an inclusion diagram. The class of *analytical sets*[1] is exactly $\bigcup_{n \geq 1} \Sigma^1_n$. An example of a non-analytical set is the set of all second-order sentences that are true in $(\mathbb{N}, +, \times)$ (the second-order theory of $(\mathbb{N}, +, \times)$).

[1]Here the notion of *analytical sets* is defined for sets of natural numbers and is not to be confused with the *analytic sets* studied in descriptive set theory [15].

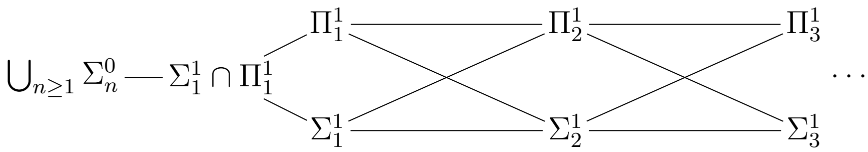

Figure 1: The analytical hierarchy

As usual in computability theory, a Gödel numbering of all finite objects of interest allows to quantify over, say, finite automata as well. We will always assume such a numbering without mentioning it explicitly.

2.2. Büchi automata

For details on Büchi automata, see [11, 26, 31]. Let Γ be a finite alphabet. With Γ^* we denote the set of all finite words over the alphabet Γ. The set of all nonempty finite words is Γ^+. An ω-word over Γ is an infinite sequence $w = a_1a_2a_3\cdots$ with $a_i \in \Gamma$. We set $w[i] = a_i$ for $i \in \mathbb{N}_+$. The set of all ω-words over Γ is denoted by Γ^ω.

A (nondeterministic) Büchi automaton is a tuple $M = (Q, \Gamma, \Delta, I, F)$, where Q is a finite set of states, $I, F \subseteq Q$ are resp. the sets of initial and final states, and $\Delta \subseteq Q \times \Gamma \times Q$ is the transition relation. If $\Gamma = \Sigma^n$ for some alphabet Σ, then we refer to M as an *n-dimensional Büchi automaton over* Σ. A *run* of M on an ω-word $w = a_1a_2a_3\cdots \in \Gamma^\omega$ is an ω-word $r = (q_1, a_1, q_2)(q_2, a_2, q_3)(q_3, a_3, q_4)\cdots \in \Delta^\omega$ such that $q_1 \in I$. The run r is *accepting* if there exists a final state from F that occurs infinitely often in r. The language $L(M) \subseteq \Gamma^\omega$ defined by M is the set of all ω-words for which there exists an accepting run. An ω-language $L \subseteq \Gamma^\omega$ is *regular* if there exists a Büchi automaton M with $L(M) = L$. The class of all regular ω-languages is effectively closed under boolean operations and projections.

For ω-words $w_1, \dots, w_n \in \Gamma^\omega$, the *convolution* $w_1 \otimes w_2 \otimes \cdots \otimes w_n \in (\Gamma^n)^\omega$ is defined by

$$w_1 \otimes w_2 \otimes \cdots \otimes w_n = (w_1[1], \dots, w_n[1])(w_1[2], \dots, w_n[2])(w_1[3], \dots, w_n[3])\cdots .$$

For $\overline{w} = (w_1, \dots, w_n)$, we write $\otimes(\overline{w})$ for $w_1 \otimes \cdots \otimes w_n$.

An n-ary relation $R \subseteq (\Gamma^\omega)^n$ is called *ω-automatic* if the ω-language $\otimes R = \{\otimes(\overline{w}) \mid \overline{w} \in R\}$ is regular, i.e., it is accepted by some n-dimensional Büchi automaton over Γ. We denote with $R(M) \subseteq (\Gamma^\omega)^n$ the relation defined by an n-dimensional Büchi-automaton over the alphabet Γ.

To also define the convolution of finite words (and of finite words with infinite words), we identify a finite word $u \in \Gamma^*$ with the ω-word $u\diamond^\omega$, where $\diamond$ is a new symbol. Then, for $u, v \in \Gamma^*, w \in \Gamma^\omega$, we write $u \otimes v$ for the ω-word $u \diamond^\omega \otimes v\diamond^\omega$ and $u \otimes w$ (resp. $w \otimes u$) for $u \diamond^\omega \otimes w$ (resp. $w \otimes u\diamond^\omega$).

2.3. ω-automatic structures

A *signature* is a finite set τ of relational symbols together with an arity $n_S \in \mathbb{N}_+$ for every relational symbol $S \in \tau$. A *τ-structure* is a tuple $\mathcal{A} = (A, (S^{\mathcal{A}})_{S\in\tau})$, where A is a non-empty set (the *universe* of $\mathcal{A}$) and $S^{\mathcal{A}} \subseteq A^{n_S}$. When the context is clear, we denote $S^{\mathcal{A}}$ with S, and we write $a \in \mathcal{A}$ for $a \in A$. Let $E \subseteq A^2$ be an equivalence relation on A. Then E is a *congruence* on $\mathcal{A}$ if $(u_1, v_1), \ldots, (u_{n_S}, v_{n_S}) \in E$ and $(u_1, \ldots, u_{n_S}) \in S$ imply $(v_1, \ldots, v_{n_S}) \in S$ for all $S \in \tau$. Then the *quotient structure* $\mathcal{A}/E$ can be defined:

- The universe of $\mathcal{A}/E$ is the set of all E-equivalence classes $[u]$ for $u \in A$.
- The interpretation of $S \in \tau$ is the relation $\{([u_1], \ldots, [u_{n_S}]) \mid (u_1, \ldots, u_{n_S}) \in S\}$.

Definition 1. *An* ω-automatic presentation *over the signature* τ *is a tuple*

$$P = (\Gamma, M, M_{\equiv}, (M_S)_{S \in \tau})$$

with the following properties:

- Γ *is a finite alphabet.*
- M *is a Büchi automaton over the alphabet* Γ*.*
- *For every* $S \in \tau$*,* M_S *is an* n_S*-dimensional Büchi automaton over the alphabet* Γ*.*
- $M_{\equiv}$ *is a 2-dimensional Büchi automaton over the alphabet* Γ *such that* $R(M_{\equiv})$ *is a congruence relation on* $(L(M), (R(M_S))_{S \in \tau})$*.*

The τ-structure defined *by the* ω*-automatic presentation* P *is the quotient structure*

$$\mathcal{S}(P) = (L(M), (R(M_S))_{S \in \tau})/R(M_{\equiv})\,.$$

If $R(M_{\equiv})$ is the identity relation on Γ^ω, then P is called *injective*. A structure $\mathcal{A}$ is *(injectively)* ω*-automatic* if there is an (injectively) ω-automatic presentation P with $\mathcal{A} \cong \mathcal{S}(P)$. In [12] it was shown that there exist ω-automatic structures that are not injectively ω-automatic. We simplify our statements by saying "given/compute an (injectively) ω-automatic structure $\mathcal{A}$" for "given/compute an (injectively) ω-automatic presentation P of a structure $\mathcal{S}(P) \cong \mathcal{A}$". *Automatic structures* [16] are defined analogously to ω-automatic structures, but instead of Büchi automata ordinary finite automata over finite words are used. For this, one has to pad shorter strings with the padding symbol $\diamond$ when defining the convolution of finite strings. More details on ω-automatic structures can be found in [3, 12, 14]. In particular, a countable structure is ω-automatic if and only if it is automatic [14].

Let $\mathsf{FO}[\exists^{\aleph_0}, \exists^{2^{\aleph_0}}]$ be first-order logic extended by the quantifiers $\exists^\kappa x \ldots$ ($\kappa \in \{\aleph_0, 2^{\aleph_0}\}$) saying that there exist exactly κ many x satisfying …. The following theorem lays out the main motivation for investigating ω-automatic structures.

Theorem 2 ([2, 14])**.** *From an* ω*-automatic presentation*

$$P = (\Gamma, M, M_{\equiv}, (M_S)_{S \in \tau})$$

and a formula $\varphi(\overline{x}) \in \mathsf{FO}[\exists^{\aleph_0}, \exists^{2^{\aleph_0}}]$ *in the signature* τ *with* n *free variables, one can compute a Büchi automaton for the relation*

$$\{(a_1, \ldots, a_n) \in L(M)^n \mid \mathcal{S}(P) \models \varphi([a_1], [a_2], \ldots, [a_n])\} .$$

In particular, the $\mathsf{FO}[\exists^{\aleph_0}, \exists^{2^{\aleph_0}}]$ *theory of any* ω*-automatic structure* $\mathcal{A}$ *is (uniformly) decidable.*

In this paper, a *graph* is a set V together with a binary relation $E \subseteq V \times V$. If every node of the graph $G = (V, E)$ has at most c successors, the graph has out-degree $\leq c$. If G has out-degree $\leq c$ for some $c \in \mathbb{N}$, then G has *finite out-degree.*

We will use the following decidability result for ω-automatic graphs; for injectively ω-automatic graphs it can be found in [4]:

Theorem 3. *It is decidable whether an* ω*-automatic graph has finite out-degree.*

Proof. Let $P = (\Gamma, M, M_\equiv, M_E)$ be an ω-automatic presentation of the graph $G = (V, E)$. We define the set

$$V^{\text{fin}} = \{(u, v) \in \Gamma^* \times \Gamma^* \mid |u| = |v|, uv^\omega \in L(M)\}$$

and the binary relations $\equiv^{\text{fin}}$ and E^{fin} on V^{fin}:

$$\begin{aligned} (u_1, v_1) \equiv^{\text{fin}} (u_2, v_2) &\iff |u_1| = |u_2| \text{ and } (u_1 v_1^\omega, u_2 v_2^\omega) \in R(M_\equiv) \\ (u_1, v_1)\, E^{\text{fin}}\, (u_2, v_2) &\iff |u_1| = |u_2| \text{ and } (u_1 v_1^\omega, u_2 v_2^\omega) \in R(M_E) \end{aligned}$$

Then it is easily seen that the graph $G^{\text{fin}} = (V^{\text{fin}}, E^{\text{fin}})/{\equiv^{\text{fin}}}$ is effectively automatic.

Let $c \in \mathbb{N}$ and define

$$\begin{aligned} L_c = \{(x, y_1, \ldots, y_c) \in L(M)^{1+c} \mid\ & \forall 1 \leq i \leq c : ([x], [y_i]) \in E \\ & \text{and } \forall 1 \leq i < j \leq c : ([y_i], [y_j]) \notin R(M_\equiv)\} . \end{aligned}$$

By Theorem 2, the relation L_c is effectively ω-automatic.

Then G does not have out-degree $< c$ iff $L_c \neq \emptyset$. Since $\otimes L_c$ is regular, this is the case iff there exists some ultimately periodic word in $\otimes L_c$, i.e., iff there are finite words $u, v, u_1, v_1, \ldots, u_c, v_c$ all of the same length with

$$(uv^\omega, u_1 v_1^\omega, \ldots, u_c v_c^\omega) \in L_c .$$

But this is equivalent to

$$\forall 1 \leq i \leq c : (u, v)\, E^{\text{fin}}\, (u_i, v_i) \quad \text{and} \quad \forall 1 \leq i < j \leq c : (u_i, v_i) \not\equiv^{\text{fin}} (u_j, v_j) .$$

Equivalently, there is a node in the automatic graph G^{fin} with at least c successors, i.e., G^{fin} does not have out-degree $< c$.

Hence G has finite out-degree iff G^{fin} has finite out-degree. Since G^{fin} is effectively automatic, this is decidable, see [20, Cor. 1] . □

Definition 4. *Let* $\mathcal{K}$ *be a class of* ω*-automatic presentations. The* isomorphism problem $\mathsf{Iso}(\mathcal{K})$ *is the set of pairs* $(P_1, P_2) \in \mathcal{K}^2$ *of* ω*-automatic presentations from* $\mathcal{K}$ *with* $\mathcal{S}(P_1) \cong \mathcal{S}(P_2)$.

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two structures over the same signature, we write $\mathcal{S}_1 \uplus \mathcal{S}_2$ for the disjoint union of the two structures. We use $\mathcal{S}^\kappa$ to denote the disjoint union of κ many copies of the structure $\mathcal{S}$, where κ is any cardinal.

The disjoint union as well as the countable or uncountable power of an automatic structure are effectively automatic, again. In this paper, we will only need this property (in a more explicit form) for injectively ω-automatic structures.

Lemma 5. *Let* $P_i = (\Gamma, M^i, M^i_{\equiv}, (M^i_S)_{S\in\tau})$ *be injectively* ω*-automatic presentations of structures* $\mathcal{S}_i$ *for* $i \in \{1, 2\}$*. One can effectively construct injectively* ω*-automatic copies of* $\mathcal{S}_1 \uplus \mathcal{S}_2$, $\mathcal{S}_1^{\aleph_0}$, *and* $\mathcal{S}_1^{2^{\aleph_0}}$ *such that*

- *The universe of the injectively* ω*-automatic copy* $\mathcal{S}$ *of* $\mathcal{S}_1 \uplus \mathcal{S}_2$ *equals* $L(M^1) \cup L(M^2)$ *and the relations are given by* $S^{\mathcal{S}} = R(M^1_S) \cup R(M^2_S)$ *provided* $L(M^1)$ *and* $L(M^2)$ *are disjoint.*
- *The universe of the injectively* ω*-automatic copy* $\mathcal{S}$ *of* $\mathcal{S}_1^{\aleph_0}$ *is* $\$^* \otimes L(M^1)$ *where* $\$$ *is a fresh symbol and the relations are given by*

$$(\$^{m_1} \otimes v_1, \ldots, \$^{m_{n_S}} \otimes v_{n_S}) \in S^{\mathcal{S}} \iff m_1 = m_2 = \ldots m_{n_S} \text{ and } (v_1, \ldots, v_{n_S}) \in S^{\mathcal{S}_1} .$$

- *The universe of the injectively* ω*-automatic copy* $\mathcal{S}$ *of* $\mathcal{S}_1^{2^{\aleph_0}}$ *is* $\{\$_1, \$_2\}^\omega \otimes L(M^1)$ *where* $\$_1$ *and* $\$_2$ *are fresh symbols and the relations are given by*

$$(u_1 \otimes v_1, \ldots, u_{n_S} \otimes v_{n_S}) \in S^{\mathcal{S}} \iff u_1 = u_2 = \ldots u_{n_S} \text{ and } (v_1, \ldots, v_{n_S}) \in S^{\mathcal{S}_1} .$$

2.4. Trees

A *forest* is a partial order $F = (V, \leq)$ such that for every $x \in V$, the set $\{y \mid y \leq x\}$ of ancestors of x is finite and linearly ordered by $\leq$. The *level* of a node $x \in V$ is $|\{y \mid y < x\}| \in \mathbb{N}$. The *height* of F is the supremum of the levels of all nodes in V; it may be infinite. Note that a forest of infinite height can be well-founded, i.e., all its paths are finite. In this paper we only deal with forests of *finite height*. For all $u \in V$, $F(u)$ denotes the restriction of F to the set $\{v \in V \mid u \leq v\}$ of successors of u. We will speak of the *subtree rooted at* u. A *tree* is a forest that has a minimal element, called the *root*. For two forests F_1, F_2 we denote with $F_1 \uplus F_2$ there disjoint union. For a set of forests $\mathcal{F}$ we write $\biguplus \mathcal{F}$ for the disjoint union of all forests in $\mathcal{F}$; it is again a forest. For a single forest F and a cardinal κ we write F^κ for the forest that consists of κ many disjoint copies of F. We use the following simple fact: Let $(T_i)_{i\in I}$ and $(U_j)_{j\in J}$ be two families of trees and let κ be an infinite cardinal which is greater than the cardinality of I and J. There may exist $i \neq j$ with $T_i \cong T_j$ and similarly for the family $(U_j)_{j\in J}$. Let the forest F (resp. G) be the disjoint union of all the T_i (resp. U_j). Then $F^\kappa \cong G^\kappa$ if and only if $(\forall i \in I\, \exists j \in J : T_i \cong U_j$

and $\forall j \in J\, \exists i \in I : T_i \cong U_j$), i.e., the two families contain the same isomorphism types of trees.

For a forest F and r not belonging to the domain of F, we denote with $r \circ F$ the tree that results from adding r to F as a new root. The *edge relation* E of the forest F is the set of pairs $(u, v) \in V^2$ such that u is the largest element in $\{x \mid x < v\}$. Note that a forest $F = (V, \leq)$ of finite height is (injectively) ω-automatic if and only if the graph (V, E) (where E is the edge relation of E) is (injectively) ω-automatic, since each of these structures is first-order interpretable in the other structure. This does not hold for trees of infinite height. For any node $u \in V$, we use $E(u)$ to denote the set of children (or immediate successors) of u.

We use $\mathcal{T}_n$ (resp. $\mathcal{T}_n^{\mathrm{i}}$) to denote the class of (injectively) ω-automatic presentations of trees of height at most n. Note that it is decidable whether a given ω-automatic presentation P belongs to $\mathcal{T}_n$ and $\mathcal{T}_n^{\mathrm{i}}$, resp., since the class of trees of height at most n can be axiomatized in first-order logic. Also the class $\bigcup_{n \geq 1} \mathcal{T}_n$ of ω-automatic presentations of trees of finite height is decidable:

Theorem 6. *For a given ω-automatic tree T, one can decide whether T has finite height.*

Proof. Let $T = (V, \leq)$. Then T has finite height if and only if there exists a constant $c \in \mathbb{N}$ such that for every $u \in V$ there are at most c many v with $v \leq u$. This is decidable by Theorem 3. □

3. ω-automatic trees of height 1 and 2

For ω-automatic trees of height 2 we need the following result:

Theorem 7 ([14])**.** *Let $\mathcal{A}$ be an ω-automatic structure and let $\varphi(x_1, \ldots, x_n, y)$ be a formula of $\mathsf{FO}[\exists^{\aleph_0}, \exists^{2^{\aleph_0}}]$. Then, for all $a_1, \ldots, a_n \in \mathcal{A}$, the cardinality of the set*

$$\{b \in \mathcal{A} \mid \mathcal{A} \models \varphi(a_1, \ldots, a_n, b)\}$$

belongs to $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$.

Theorem 8. *The following holds:*

- *The isomorphism problem $\mathsf{Iso}(\mathcal{T}_1)$ for ω-automatic trees of height 1 is decidable.*
- *There exists a tree U such that $\{P \in \mathcal{T}_2^{\mathrm{i}} \mid \mathcal{S}(P) \cong U\}$ is Π_1^0-hard. The isomorphism problems $\mathsf{Iso}(\mathcal{T}_2)$ and $\mathsf{Iso}(\mathcal{T}_2^{\mathrm{i}})$ for (injectively) ω-automatic trees of height 2 are Π_1^0-complete.*

Proof. Two trees of height 1 are isomorphic if and only if they have the same size. By Theorem 7, the number of elements in an ω-automatic tree $\mathcal{S}(P)$ with $P \in \mathcal{T}_1$ is either finite, $\aleph_0$ or $2^{\aleph_0}$ and the exact size can be computed using Theorem 2 (by checking successively validity of the sentences $\exists^\kappa x : x = x$ for $\kappa \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$[2]).

[2]Where $\exists^n x : \varphi(x)$ for $n \in \mathbb{N}$ is shorthand for the obvious first-order formula expressing that there are exactly n elements satisfying φ.

By [21], there is a countable tree U of height 2 such that the set of automatic presentations of U is Π^0_1-hard. Since, from an automatic presentation P' one can construct an injectively ω-automatic presentation P with $\mathcal{S}(P') \cong \mathcal{S}(P)$, the set of (injectively) ω-automatic presentations of U (and therefore the isomorphism problem for (injectively) ω-automatic trees of height at most 2) is Π^0_1-hard as well.

To show containment in Π^0_1, let us take two trees T_1 and T_2 of height 2 and let E_i be the edge relation of T_i and r_i its root. For $i \in \{1, 2\}$ and a cardinal λ let $\kappa_{\lambda,i}$ be the cardinality of the set of all $u \in E_i(r_i)$ such that $|E_i(u)| = \lambda$. Then $T_1 \cong T_2$ if and only if $\kappa_{\lambda,1} = \kappa_{\lambda,2}$ for any cardinal λ. Now assume that T_1 and T_2 are both ω-automatic. By Theorem 7, for all $i \in \{1, 2\}$ and every $u \in E_i(r_i)$ we have $|E_i(u)| \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$. Moreover, again by Theorem 7, every cardinal $\kappa_{\lambda,i}$ ($\lambda \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$, $i \in \{1, 2\}$) belongs to $\mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ as well. Hence, $T_1 \cong T_2$ if and only if for all $\kappa, \lambda \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$:

$$\begin{aligned} & T_1 \models \exists^\kappa x : ((r_1, x) \in E \wedge \exists^\lambda y : (x, y) \in E) \\ \text{if and only if} \quad & T_2 \models \exists^\kappa x : ((r_2, x) \in E \wedge \exists^\lambda y : (x, y) \in E) \,. \end{aligned}$$

By Theorem 2, this equivalence is decidable for all κ, λ. Since it has to hold for all κ, λ, the isomorphism of two ω-automatic trees of height 2 is expressible by a Π^0_1-statement. □

4. A normal form for analytical sets

To prove our lower bound for the isomorphism problem of ω-automatic trees of height $n \geq 3$, we will use the following normal form for analytical sets. A formula of the form $x \in X$ or $x \notin X$ is called a *set constraint*. The constructions in the following proof are standard.

Proposition 9. *For every odd (resp. even) $n \in \mathbb{N}_+$ and every Π^1_n (resp. Σ^1_n) relation $A \subseteq \mathbb{N}^r_+$, there exist polynomials $p_i, q_i \in \mathbb{N}[\overline{x}, y, \overline{z}]$ and disjunctions ψ_i $(1 \leq i \leq \ell)$ of set constraints (on the set variables $X_1, \ldots, X_n$ and individual variables $\overline{x}, y, \overline{z}$) such that $\overline{x} \in A$ if and only if*

$$Q_1 X_1 \; Q_2 X_2 \cdots Q_n X_n \; \exists y \; \forall \overline{z} : \bigwedge_{i=1}^{\ell} p_i(\overline{x}, y, \overline{z}) \neq q_i(\overline{x}, y, \overline{z}) \vee \psi_i(\overline{x}, y, \overline{z}, X_1, \ldots, X_n),$$

where $Q_1, Q_2, \ldots, Q_n$ are alternating quantifiers with $Q_n = \forall$. Moreover, if the Π^1_n (resp. Σ^1_n) relation $A \subseteq \mathbb{N}^r_+$ is given by a second-order formula as in (1), then the polynomials p_i, q_i and the disjunctions ψ_i can be effectively computed.

Proof. For notational simplicity, we present the proof only for the case when n is odd. The other case can be proved in a similar way by just adding an existential quantification $\exists X_0$ at the beginning. We will write $\Sigma_m(\mathsf{SC}, \mathsf{REC})$ for the set of first-order Σ_m-formulas over set constraints and recursive predicates, where all quantifiers range over $\mathbb{N}_+$. The set $\Pi_m(\mathsf{SC}, \mathsf{REC})$ is to be understood similarly and $B\Sigma_m(\mathsf{SC}, \mathsf{REC})$ is the set of boolean combinations of formulas from $\Sigma_m(\mathsf{SC},\mathsf{REC})$. With $C_k : \mathbb{N}_+^k \to \mathbb{N}_+$ we will denote some computable bijection.

Fix an odd number n. It is well known that every Π^1_n-relation $A \subseteq \mathbb{N}_+^r$ can be written as

$$A = \{\overline{x} \in \mathbb{N}_+^r \mid \forall f_1\ \exists f_2 \cdots \forall f_n\ \exists y : P(\overline{x}, y, f_1, \ldots, f_n)\}, \tag{2}$$

where P is a recursive predicate relative to the functions $f_1, \ldots, f_n$ (see [25, p. 378]). In other words, there exists an oracle Turing-machine which computes the boolean value $P(\overline{x}, y, f_1, \ldots, f_n)$ from input $(\overline{x}, y)$. The oracle Turing-machine can compute a value $f_i(a)$ for a previously computed number $a \in \mathbb{N}_+$ in a single step. Therefore we can easily obtain an oracle Turing-machine M which halts on input $\overline{x}$ if and only if $\exists y : P(\overline{x}, y, f_1, \ldots, f_n)$ holds.

Following [25], we can replace the function quantifiers in (2) by set quantifiers as follows. A function $f : \mathbb{N}_+ \to \mathbb{N}_+$ is encoded by the set $\{C_2(x, y) \mid f(x) = y\}$. Let $\mathsf{func}(X)$ be the following formula, where X is a set variable:

$$\begin{aligned}\mathsf{func}(X) \quad = \quad & (\forall x, y, z, u, v : C_2(x, y) = u \wedge C_2(x, z) = v \wedge u, v \in X \to y = z) \wedge \\ & (\forall x\ \exists y, z : C_2(x, y) = z \wedge z \in X)\end{aligned}$$

Hence, $\mathsf{func}(X)$ is a $\Pi_2(\mathsf{SC},\mathsf{REC})$-formula, which expresses that X encodes a total function on $\mathbb{N}_+$. Then, the set A in (2) can be defined by the formula

$$\forall X_1 : \neg\mathsf{func}(X_1) \vee \exists X_2 : \mathsf{func}(X_2) \wedge \cdots \forall X_n : \neg\mathsf{func}(X_n) \vee R(\overline{x}, X_1, \ldots, X_n). \tag{3}$$

The predicate R can be derived from the oracle Turing-machine M as follows: Construct from M a new oracle Turing-machine N with oracle sets $X_1, \ldots, X_n$. If the machine M wants to compute the value $f_i(a)$, then the machine N starts to enumerate all $b \in \mathbb{N}_+$ until it finds $b \in \mathbb{N}_+$ with $C_2(a, b) \in X_i$. Then it continues its computation with b for $f_i(a)$. Then the predicate $R(\overline{x}, X_1, \ldots, X_n)$ expresses that machine N halts on input $\overline{x}$.

Fix a computable bijection $D : \mathbb{N}_+ \to \mathsf{Fin}(\mathbb{N}_+)$, where $\mathsf{Fin}(\mathbb{N}_+)$ is the set of all finite subsets of $\mathbb{N}_+$. Let $\mathsf{in}(x, y)$ be an abbreviation for $x \in D(y)$. This is a computable predicate.

Next, consider the predicate $R(\overline{x}, X_1, \ldots, X_n)$. In every terminating run of the machine N on input $\overline{x}$, the machine N makes only finitely many oracle queries. Hence, the predicate $R(\overline{x}, X_1, \ldots, X_n)$ is equivalent to

$$\exists b\ \exists(s_1, \ldots, s_n) : S(\overline{x}, b, (s_1, \ldots, s_n)) \wedge \bigwedge_{i=1}^{n} \forall z \leq b\ (\mathsf{in}(z, s_i) \leftrightarrow z \in X_i),$$

where the predicate S is derived from the Turing-machine N as follows: Let T be the Turing-machine that on input $(\overline{x}, b, (s_1, \ldots, s_n))$ behaves as N, but if N asks the oracle whether $z \in X_i$, then T first checks whether $z \leq b$ (if not, then T diverges) and then checks, whether $\mathsf{in}(z, s_i)$ holds. Then $S(\overline{x}, b, (s_1, \ldots, s_n))$ if and only if T halts on input $(\overline{x}, b, (s_1, \ldots, s_n))$. Hence, the predicate $S(\overline{x}, b, (s_1, \ldots, s_n))$ is recursively enumerable, i.e., can be described by a formula from $\Sigma_1(\mathsf{SC},\mathsf{REC})$. Hence the predicate R can be described by a formula from $\Sigma_2(\mathsf{SC},\mathsf{REC})$.

Note that the formula from (3) is equivalent with a formula

$$\forall X_1 \exists X_2 \cdots \forall X_n : \varphi(\overline{x}, \overline{X}), \tag{4}$$

where φ is a boolean combination of R and formulas of the form $\mathsf{func}(X_i)$. Since all these formulas belong to $\Pi_2(\mathsf{SC}, \mathsf{REC}) \cup \Sigma_2(\mathsf{SC}, \mathsf{REC})$, the formula φ belongs to $B\Sigma_2(\mathsf{SC}, \mathsf{REC}) \subseteq \Pi_3(\mathsf{SC}, \mathsf{REC})$. Hence (4) is equivalent to

$$\forall X_1\ \exists X_2 \cdots \forall X_n \forall \overline{a}\ \exists \overline{b}\ \forall \overline{c} : \beta, \tag{5}$$

where β is a boolean combination of recursive predicates and set constraints.

We can eliminate the quantifier block $\forall \overline{a}$ by merging it with $\forall X_n$: First, we can reduce $\forall \overline{a}$ to a single quantifier $\forall a$. For this, assume that the length of the tuple $\overline{a}$ is k. Then, $\forall \overline{a} \cdots$ in (5) can be replaced by $\forall a\ \exists \overline{a} : C_k(\overline{a}) = a \wedge \cdots$. Since $C_k(\overline{a}) = a$ is again recursive and since we can merge $\exists \overline{a}\, \exists \overline{b}$ into a single block of quantifiers $\exists \overline{b}$, we obtain indeed an equivalent formula of the form

$$\forall X_1\ \exists X_2 \cdots \forall X_n\ \forall a\ \exists \overline{b}\ \forall \overline{c} : \beta', \tag{6}$$

where β' is a boolean combination of recursive predicates and set constraints.

Next, we encode the pair (X_n, a) by the set $\{2x \mid x \in X_n\} \cup \{2a+1\}$. Let $\alpha(X)$ be the formula

$$\begin{aligned}\alpha(X) \quad = \quad & (\forall x, y, x', y' : x = 2x' + 1 \wedge y = 2y' + 1 \wedge x, y \in X \ \rightarrow \ x = y) \wedge \\ & (\exists x, u : x \in X \wedge x = 2u + 1).\end{aligned}$$

Hence, $\alpha(X)$ expresses that X contains exactly one odd number. Hence, we obtain a formula equivalent to (6) by

- replacing $\forall X_n\ \forall a \cdots$ with $\forall X_n : \neg\alpha(X_n) \vee \exists a, a' : a' \in X_n \wedge a' = 2a + 1 \wedge \cdots$ and
- replacing every existential quantifier $\exists b_i \cdots$ (resp. universal quantifier $\forall c_i \cdots$) in (6) with $\exists b_i\ \exists b_i' : b_i' = 2b_i \wedge \cdots$ (resp. $\forall c_i\ \forall c_i' : c_i' \neq 2c_i \vee \cdots$), and
- replacing every sub-formula $a \in X_n$, $b_i \in X_n$, or $c_i \in X_n$ with $a' \in X_n$, $b_i' \in X_n$, or $c_i' \in X_n$, resp..

All new quantifiers can be merged with either the block $\exists \overline{b}$ or the block $\forall \overline{c}$ in (6). We now have obtained an equivalent formula of the form

$$\forall X_1\ \exists X_2 \cdots \forall X_n\ \exists \overline{b}\ \forall \overline{c} : \beta'', \tag{7}$$

where β'' is a boolean combination of recursive predicates and set constraints.

The block $\exists \overline{b} \cdots$ can be replaced by $\exists b\ \forall \overline{b} : C_\ell(\overline{b}) \neq b \vee \cdots$, where ℓ is the length of the tuple $\overline{b}$. Since $C_\ell(\overline{b}) \neq b$ is a computable predicate, this results in an equivalent formula of the form

$$\forall X_1\ \exists X_2 \cdots \forall X_n\ \exists b\ \forall \overline{c} : \beta''',$$

where β''' is a boolean combination of recursive predicates and set constraints.

Note that the set of recursive predicates is closed under boolean combinations and that the set of set constraints is closed under negation. This allows to obtain an equivalent formula of the form

$$\forall X_1 \, \exists X_2 \cdots \forall X_n \, \exists b \, \forall \overline{c} : \bigwedge_{i=1}^{\ell} (R_i \vee \psi_i),$$

where the R_i are recursive predicates and the ψ_i are disjunctions of set constraints.

Since the recursive predicates R_i are co-Diophantine (cf. [24]), there are polynomials $p_i, q_i \in \mathbb{N}[b, \overline{c}, \overline{z}]$ such that $R_i(b, \overline{c})$ is equivalent to $\forall \overline{z} : p_i(b, \overline{c}, \overline{z}) \neq q_i(b, \overline{c}, \overline{z})$. Replacing R_i in the above formula by this equivalent formula and merging the new universal quantifiers $\forall \overline{z}$ with $\forall \overline{c}$ results in a formula as required.

Since all the steps in our construction can be made effective, the second part of the proposition concerning effectiveness follows. □

It is known that the first-order quantifier block $\exists y \, \forall \overline{z}$ in Proposition 9 cannot be replaced by a block with only one type of first-order quantifiers, see e.g. [25, p. 379].

5. ω-automatic trees of height at least 4

Our main technical result for injectively ω-automatic trees of height at least 4, whose proof occupies Sections 5.1 and 5.2, is the following:

Proposition 10. *From a given $n \geq 1$, one can compute injectively ω-automatic trees $U[0]$ and $U[1]$ of height $n+3$ such that the following holds: From a given set $A \subseteq \mathbb{N}_+$ that is Π^1_n if n is odd and Σ^1_n if n is even*[3] *and a given $x \in \mathbb{N}_+$ one can compute an injectively ω-automatic tree $T[x]$ of height $n+3$ with $T[x] \cong U[0]$ if and only if $x \in A$ and $T[x] \cong U[1]$ otherwise.*

Before we prove Proposition 10, let us first state some easy consequences.

Corollary 11. *From given $n \geq 1$ and $\Theta \in \{\Sigma, \Pi\}$, one can compute an injectively ω-automatic tree $U_{n,\Theta}$ of height $n+3$ such that the set $\{P \in \mathcal{T}^{\mathrm{i}}_{n+3} \mid \mathcal{S}(P) \cong U_{n,\Theta}\}$ is hard for Θ^1_n.*

Proof. Let $n \geq 1$ be odd. Let A be an arbitrary set from Π^1_n and set $U_{n,\Pi} = U[0]$ and $U_{n,\Sigma} = U[1]$. Then the mapping $x \mapsto T[x]$ is a reduction from A to $\{P \in \mathcal{T}^{\mathrm{i}}_{n+3} \mid \mathcal{S}(P) \cong U_{n,\Pi}\}$ and, at the same time, a reduction from the Σ^1_n-set $\mathbb{N}_+ \setminus A$ to $\{P \in \mathcal{T}^{\mathrm{i}}_{n+3} \mid \mathcal{S}(P) \cong U_{n,\Sigma}\}$. Since A was chosen arbitrary from Π^1_n, the statement follows for n odd. If n is even, we can proceed similarly exchanging the roles of $U[0]$ and $U[1]$. □

Corollary 12. *The following holds for all $n \geq 1$:*

[3] It is assumed that the set A is given by a second-order formula as in (1).

- *The isomorphism problem* $\mathsf{Iso}(\mathcal{T}^{\mathrm{i}}_{n+3})$ *for the class of injectively* ω*-automatic trees of height* $n+3$ *is hard for both the classes* Π^1_n *and* Σ^1_n.
- *The second-order theory of* $(\mathbb{N},+,\times)$ *can be reduced to the isomorphism problem* $\mathsf{Iso}(\bigcup_{n\geq 1}\mathcal{T}^{\mathrm{i}}_n)$ *for the class of all injectively* ω*-automatic trees of finite height. Hence, the isomorphism problem* $\mathsf{Iso}(\bigcup_{n\geq 1}\mathcal{T}^{\mathrm{i}}_n)$ *is not analytical.*

We now start to prove Proposition 10. Let A be a set (given by a second-order formula) that is Π^1_n if n is odd and Σ^1_n otherwise. By Proposition 9 it can be written effectively in the form

$$A = \{x \in \mathbb{N}_+ \mid Q_1X_1 \cdots Q_nX_n \exists y\, \forall \overline{z} : \bigwedge_{i=1}^{\ell} p_i(x,y,\overline{z}) \neq q_i(x,y,\overline{z}) \vee \psi_i(x,y,\overline{z},\overline{X})\}, \tag{8}$$

where

- $Q_1, Q_2, \ldots, Q_n$ are alternating quantifiers with $Q_n = \forall$,
- p_i, q_i $(1 \leq i \leq \ell)$ are polynomials in $\mathbb{N}[x,y,\overline{z}]$ where $\overline{z}$ has length k, and
- every ψ_i $(1 \leq i \leq \ell)$ is a disjunction of set constraints on the set variables $X_1, \ldots, X_n$ and the individual variables $x, y, \overline{z}$.

Let $\varphi_{-1}(x,y,X_1,\ldots,X_n)$ be the formula

$$\forall \overline{z} : \bigwedge_{i=1}^{\ell} p_i(x,y,\overline{z}) \neq q_i(x,y,\overline{z}) \vee \psi_i(x,y,\overline{z},\overline{X}) \,.$$

For $0 \leq m \leq n$, we will also consider the formula $\varphi_m(x, X_1, \ldots, X_{n-m})$ defined by

$$Q_{n+1-m}X_{n+1-m} \cdots Q_nX_n\ \exists y : \varphi_{-1}(x,y,X_1,\ldots,X_n)$$

such that $\varphi_0(x,X_1,\ldots,X_n)$ is a first-order formula and $\varphi_n(x)$ holds if and only if $x \in A$.

To prove Proposition 10, we construct by induction on $0 \leq m \leq n$ height-$(m+3)$ trees $T_m[X_1,\ldots,X_{n-m},x]$ and $U_m[i]$ where $X_1,\ldots,X_{n-m} \subseteq \mathbb{N}_+$, $x \in \mathbb{N}_+$, and $i \in \{0,1\}$ such that the following holds:

$$\forall \overline{X} \in (2^{\mathbb{N}_+})^{n-m}\ \forall x \in \mathbb{N}_+ : T_m[\overline{X},x] \cong \begin{cases} U_m[0] & \text{if } \varphi_m(x,\overline{X}) \text{ holds} \\ U_m[1] & \text{otherwise} \end{cases} \tag{9}$$

Setting $T[x] = T_n[x]$, $U[0] = U_n[0]$, and $U[1] = U_n[1]$ and effectively constructing from x, n, and the formula for the set A injectively ω-automatic presentations for $T[x]$, $U[0]$, and $U[1]$ then proves Proposition 10.

5.1. Construction of trees

In the following, we will use the *injective* polynomial function

$$C : \mathbb{N}_+^2 \to \mathbb{N}_+ \text{ with } C(x,y) = (x+y)^2 + 3x + y. \tag{10}$$

For $e_1, e_2 \in \mathbb{N}_+$, let $S[e_1, e_2]$ denote the height-1 tree containing $C(e_1, e_2)$ leaves. For $(\overline{X}, x, y, \overline{z}, z_{k+1}) \in (2^{\mathbb{N}_+})^n \times \mathbb{N}_+^{k+3}$ and $1 \le i \le \ell$, define the following height-1 tree, where ℓ, p_i, q_i, and ψ_i refer to the definition of the set A above:[4]

$$T'[\overline{X}, x, y, \overline{z}, z_{k+1}, i] = \begin{cases} S[1,2] & \text{if } \psi_i(x, y, \overline{z}, \overline{X}) \\ S[p_i(x,y,\overline{z}) + z_{k+1}, q_i(x,y,\overline{z}) + z_{k+1}] & \text{otherwise.} \end{cases} \tag{11}$$

Next, we define the following height-2 trees, where $\kappa \in \mathbb{N}_+ \cup \{\omega\}$ (we consider the natural order on $\mathbb{N}_+ \cup \{\omega\}$ with $n < \omega$ for all $n \in \mathbb{N}_+$):

$$T''[\overline{X}, x, y] = r \circ \begin{pmatrix} \biguplus \{S[e_1, e_2] \mid e_1 \neq e_2\} \uplus \\ \biguplus \{T'[\overline{X}, x, y, \overline{z}, z_{k+1}, i] \mid \overline{z} \in \mathbb{N}_+^k, z_{k+1} \in \mathbb{N}_+, 1 \le i \le \ell\} \end{pmatrix}^{\aleph_0} \tag{12}$$

$$U''[\kappa] = r \circ \Big(\biguplus \{S[e_1, e_2] \mid e_1 \neq e_2\} \uplus \biguplus \{S[e,e] \mid \kappa \le e < \omega\} \Big)^{\aleph_0}. \tag{13}$$

Note that all the trees $T''[\overline{X}, x, y]$ and $U''[\kappa]$ are build from trees of the form $S[e_1, e_2]$. Furthermore, if $S[e,e]$ appears as a building block, then $S[e+a, e+a]$ also appears as one for all $a \in \mathbb{N}$ (this is the reason for introducing the additional variable z_{k+1} in (11)). In addition, any building block $S[e_1, e_2]$ appears either $\aleph_0$ many times or not at all. In this sense, $U''[\kappa]$ encodes the set

$$\{(e_1, e_2) \mid e_1 \neq e_2\} \cup \{(e,e) \mid \kappa \le e < \omega\}$$

and $T''[\overline{X}, x, y]$ encodes the set

$$\{(e_1, e_2) \mid e_1 \neq e_2\} \cup \{(p_i(x,y,\overline{z}) + z_{k+1}, q_i(x,y,\overline{z}) + z_{k+1}) \mid \; 1 \le i \le \ell, z_{k+1} \in \mathbb{N}_+, \overline{z} \in \mathbb{N}_+^k, \psi_i(x,y,\overline{z},\overline{X}) \text{ does not hold}\}.$$

These observations allow to prove the following:

Lemma 13. *Let $\overline{X} \in (2^{\mathbb{N}_+})^n$ and $x, y \in \mathbb{N}_+$. Then the following hold:*

(a) $T''[\overline{X}, x, y] \cong U''[\kappa]$ for some $\kappa \in \mathbb{N}_+ \cup \{\omega\}$

(b) $T''[\overline{X}, x, y] \cong U''[\omega]$ if and only if $\varphi_{-1}(x, y, \overline{X})$ holds

[4]The choice of $S[1,2]$ in the first case is arbitrary. Any $S[a,b]$ with $a \neq b$ would work.

Proof. Let us start with property (b). Suppose $\varphi_{-1}(x,y,\overline{X})$ holds. Let $\overline{z} \in \mathbb{N}_+^k$, $z_{k+1} \in \mathbb{N}$, and $1 \le i \le \ell$. Then $p_i(x,y,\overline{z}) \neq q_i(x,y,\overline{z})$ or $\psi_i(x,y,\overline{z},\overline{X})$ holds. In any case, there are natural numbers $e_1 \neq e_2$ with $T'[\overline{X},x,y,\overline{z},z_{k+1},i] = S[e_1,e_2]$. Hence $T''[\overline{X},x,y] \cong U''[\omega]$.

Conversely, suppose we have $T''[\overline{X},x,y] \cong U''[\omega]$. Let $\overline{z} \in \mathbb{N}^k$, $z_{k+1} \in \mathbb{N}$, and $1 \le i \le \ell$. Then the tree $T'[\overline{X},x,y,\overline{z},z_{k+1},i]$ is a height-1 subtree of $T''[\overline{X},x,y] \cong U''[\omega]$. This means that there are natural numbers $e_1 \neq e_2$ with $T'[\overline{X},x,y,\overline{z},z_{k+1},i] \cong S[e_1,e_2]$. By (11), this implies $p_i(x,y,\overline{z}) \neq q_i(x,y,\overline{z}) \vee \psi_i(x,y,\overline{z},\overline{X})$. Hence the formula

$$\forall \overline{z} : \bigwedge_{i=1}^{\ell} p_i(x,y,\overline{z}) \neq q_i(x,y,\overline{z}) \vee \psi_i(x,y,\overline{z},\overline{X})$$

holds.

Now it suffices to prove statement (a) in case $\varphi_{-1}(x,y,\overline{X})$ does not hold. Then there exist some $\overline{z} \in \mathbb{N}_+^k$ and $1 \le i \le \ell$ with

$$p_i(x,y,\overline{z}) = q_i(x,y,\overline{z}) \wedge \neg\psi_i(x,y,\overline{z},\overline{X})\,.$$

Hence there is some $e \in \mathbb{N}_+$ such that $S[e,e]$ appears in the definition of $T''[\overline{X},x,y]$. Let $m = \min\{e \in \mathbb{N}_+ \mid S[e,e] \text{ appears in } T''[\overline{X},x,y]\}$. Then, for all $a \in \mathbb{N}$, also $S[m+a,m+a]$ appears in $T''[\overline{X},x,y]$. Hence $T''[\overline{X},x,y] \cong U''[m]$. □

In a next step, we collect the trees $T''[\overline{X},x,y]$ and $U''[\kappa]$ into the trees $T_0[\overline{X},x]$, $U_0[0]$, and $U_0[1]$ as follows:

$$T_0[\overline{X},x] = r \circ \Big(\biguplus\{U''[m] \mid m \in \mathbb{N}_+\} \uplus \biguplus\{T''[\overline{X},x,y] \mid y \in \mathbb{N}_+\}\Big)^{\aleph_0} \tag{14}$$

$$U_0[0] = r \circ \Big(\biguplus\{U''[\kappa] \mid \kappa \in \mathbb{N}_+ \cup \{\omega\}\}\Big)^{\aleph_0} \tag{15}$$

$$U_0[1] = r \circ \Big(\biguplus\{U''[m] \mid m \in \mathbb{N}_+\}\Big)^{\aleph_0} \tag{16}$$

By Lemma 13(a), these trees are build from copies of the trees $U''[\kappa]$ (and are therefore of height 3), each appearing either $\aleph_0$ many times or not at all. The following lemma states (9) for $m = 0$:

Lemma 14. *Let $\overline{X} \in (2^{\mathbb{N}_+})^n$ and $x \in \mathbb{N}_+$. Then*

$$T_0[\overline{X},x] \cong \begin{cases} U_0[0] & \text{if } \varphi_0(x,\overline{X}) \text{ holds and} \\ U_0[1] & \text{otherwise.} \end{cases}$$

Proof. If $T_0[\overline{X},x] \cong U_0[0]$, then there must be some $y \in \mathbb{N}_+$ such that $T''[\overline{X},x,y] \cong U''[\omega]$. By Lemma 13(b), this means that $\varphi_0(x,\overline{X})$ holds.

On the other hand, suppose $T_0[\overline{X},x] \not\cong U_0[0]$. Then $T''[\overline{X},x,y] \not\cong U''[\omega]$ for all $y \in \mathbb{N}_+$. From Lemma 13(b) again, we obtain for all $y \in \mathbb{N}_+$: $T''[\overline{X},x,y] \cong U''[m_y]$ for some $m_y \in \mathbb{N}_+$. Hence $T_0[\overline{X},x] \cong U_0[1]$ in this case. □

Now, we come to the induction step in the construction of our trees. Suppose that for some $0 \leq m < n$ we have height-$(m+3)$ trees $T_m[X_1, \ldots, X_{n-m}, x]$, $U_m[0]$ and $U_m[1]$ satisfying (9). Let $\overline{X}$ stand for $(X_1, \ldots, X_{n-m-1})$ and let $\alpha = m \bmod 2$. We define the following height-$(m+4)$ trees:

$$T_{m+1}[\overline{X}, x] \;=\; r \circ \Big(U_m[\alpha] \uplus \biguplus \{ T_m[\overline{X}, X_{n-m}, x] \mid X_{n-m} \subseteq \mathbb{N}_+ \} \Big)^{2^{\aleph_0}} \tag{17}$$

$$U_{m+1}[i] \;=\; r \circ (U_m[\alpha] \uplus U_m[i])^{2^{\aleph_0}} \quad \text{for } i \in \{0,1\} \tag{18}$$

Note that the trees $T_{m+1}[\overline{X}, x]$, $U_{m+1}[0]$, and $U_{m+1}[1]$ consist of $2^{\aleph_0}$ many copies of $U_m[\alpha]$ and possibly $2^{\aleph_0}$ many copies of $U_m[1-\alpha]$.

Lemma 15. *Let $X_1, \ldots, X_{n-m-1} \subseteq \mathbb{N}_+$ and $x \in \mathbb{N}_+$. Then*

$$T_{m+1}[X_1, \ldots, X_{n-m-1}, x] \cong \begin{cases} U_{m+1}[0] & \textit{if } \varphi_{m+1}(x, X_1, \ldots X_{n-m-1}) \textit{ holds} \\ U_{m+1}[1] & \textit{otherwise.} \end{cases}$$

Proof. We have to handle the cases of odd and even m separately and start assuming m to be even (i.e., $\alpha = 0$) such that the outermost quantifier Q_{n-m} of the formula $\varphi_{m+1}(x, X_1, \ldots, X_{n-m-1})$ is universal.

Suppose that $\varphi_{m+1}(X_1, \ldots, X_{n-m-1}, x)$ holds. Then, by the inductive hypothesis, for each $X_{n-m} \subseteq \mathbb{N}_+$, $T_m[X_1, \ldots, X_{n-m}, x] \cong U_m[0]$. Hence all height-$(m+3)$ subtrees of $T_{m+1}[X_1, \ldots, X_{n-m-1}, x]$ are isomorphic to $U_m[0]$ and thus

$$T_{m+1}[X_1, \ldots, X_{n-m-1}, x] \cong r \circ U_m[0]^{2^{\aleph_0}} = U_{m+1}[0]\,.$$

On the other hand, suppose that $\neg\varphi_{m+1}(X_1, \ldots, X_{n-m-1}, x)$ holds. Then there exists some set X_{n-m} such that $\neg\varphi_m(X_1, \ldots, X_{n-m}, x)$ is true. Hence, by the induction hypothesis,

$$T_m(X_1, \ldots, X_{n-m}, x) \cong U_m[1],$$

i.e., $T_{m+1}(X_1, \ldots, X_{n-m-1}, x)$ contains one (and therefore $2^{\aleph_0}$ many) height-$(m+3)$ subtrees isomorphic to $U_m[1]$. This implies $T_{m+1}(X_1, \ldots, X_{n-m-1}, x) \cong U_{m+1}[1]$ since m is even.

The arguments for m odd are very similar and therefore left to the reader. □

The following lemma follows from Lemma 15 with $m = n-1$ and the fact that $\varphi_n(x)$ holds if and only if $x \in A$.

Lemma 16. *For all $x \in \mathbb{N}_+$, we have $T_n[x] \cong U_n[0]$ if $x \in A$ and $T_n[x] \cong U_n[1]$ otherwise.*

5.2. Injective ω-automaticity

Injectively ω-automatic presentations of the trees $T_m[\overline{X}, x]$, $U_m[0]$, and $U_m[1]$ will be constructed inductively. Note that the construction of $T_{m+1}[\overline{X}, x]$ involves all the trees $T_m[\overline{X}, X_{n-m}, x]$ for $X_{n-m} \subseteq \mathbb{N}_+$. Hence we need *one single injectively ω-automatic presentation* for the forest consisting of all these trees. Therefore, we will deal with forests. We will prove Lemma 17, for which we need the following definitions.

Let 0, 1, a, and b be symbols. For an ω-language L, we write $\otimes_k(L)$ for $\otimes(L^k)$. For $X \subseteq \mathbb{N}_+$, let $w_X \in \{0,1\}^\omega$ be the characteristic word (i.e., $w_X[i] = 1$ if and only if $i \in X$) and, for $\overline{X} = (X_1, \dots, X_n) \in (2^{\mathbb{N}_+})^n$, write $w_{\overline{X}}$ for the convolution of the words w_{X_i}.

Lemma 17. *From each $0 \le m \le n$, one can effectively construct an injectively ω-automatic forest $\mathcal{H}_m$ such that*

- *the set of roots of $\mathcal{H}_m$ is $\big(\otimes_{n-m}(\{0,1\}^\omega) \otimes a^+\big) \cup \{\varepsilon, b\}$,*
- *$\mathcal{H}_m(w_{\overline{X}} \otimes a^x) \cong T_m[\overline{X}, x]$ for all $\overline{X} \in (2^{\mathbb{N}_+})^{n-m}$ and $x \in \mathbb{N}_+$,*
- *$\mathcal{H}_m(\varepsilon) \cong U_m[0]$, and*
- *$\mathcal{H}_m(b) \cong U_m[1]$.*

Before we prove Lemma 17, let us first deduce Proposition 10 (and therefore Corollary 11). Note that $T_n[x]$ is the tree in $\mathcal{H}_n$ rooted at a^x. Hence $T_n[x]$ is (effectively) an injectively ω-automatic tree. Now Lemma 16 finishes the proof of Proposition 10.

We will construct the forest $\mathcal{H}_{m+1}$ from $\mathcal{H}_m$ by the following general strategy: Add a set of new roots to $\mathcal{H}_m$ and connect them to some of the old roots *which results in a directed acyclic graph* (or dag) and not necessarily in a forest. The forest $\mathcal{H}_m$ will then be the unfolding of this dag.

The *height* of a dag D is the length (number of edges) of a longest directed path in D. We only consider dags of finite height. A *root* of a dag is a node without incoming edges. A dag $D = (V, E)$ can be unfolded into a forest $\mathrm{unfold}(D)$ in the usual way: Nodes of $\mathrm{unfold}(D)$ are directed paths in D that start in a root and the order relation is the prefix relation between these paths. For a root $v \in V$ of D, we define the tree $\mathrm{unfold}(D, v)$ as the restriction of $\mathrm{unfold}(D)$ to those paths that start in v. We will make use of the following lemma whose proof is based on the immediate observation that the set of convolutions of paths in D is again a regular ω-language.

Lemma 18. *From a given $k \in \mathbb{N}$ and an injectively ω-automatic presentation for a dag D of height at most k, one can construct effectively an injectively ω-automatic presentation for* $\mathrm{unfold}(D)$ *such that the roots of* $\mathrm{unfold}(D)$ *coincide with the roots of D and* $\mathrm{unfold}(D, r) = (\mathrm{unfold}(D))(r)$ *for any root r.*

Proof. Let $D = (V, E) = \mathcal{S}(P)$, i.e., V is an ω-regular language and the binary relation $E \subseteq V \times V$ is ω-automatic. The universe for our injectively ω-automatic copy of $\mathrm{unfold}(D)$ is the set L of all convolutions $v_0 \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_m$, where v_0 is a root and $(v_i, v_{i+1}) \in E$ for all $0 \le i < m$. Since the dag D has height at most k, we have $m \le k$. Since the edge relation of D is ω-automatic and since the set of all roots in D is FO-definable and hence ω-regular by Theorem 2, L is indeed an ω-regular set. Moreover, the edge relation of $\mathrm{unfold}(D)$ becomes clearly ω-automatic on L. $\square$

For a symbol a and a tuple $\overline{e} = (e_1, \dots, e_k) \in \mathbb{N}_+^k$, we write $a^{\overline{e}}$ for the ω-word

$$a^{e_1} \otimes a^{e_2} \otimes \cdots \otimes a^{e_k} = (a^{e_1} \diamond^\omega) \otimes (a^{e_2} \diamond^\omega) \otimes \cdots \otimes (a^{e_k} \diamond^\omega) .$$

The following lemma was shown in [21] for finite words instead of ω-words.

Lemma 19. *Given a non-zero polynomial $p(\overline{x}) \in \mathbb{N}[\overline{x}]$ in k variables, one can effectively construct a Büchi automaton $\mathcal{B}[p(\overline{x})]$ over the alphabet $\{a, \diamond\}^k$ with $L(\mathcal{B}[p(\overline{x})]) = \otimes_k(a^+)$ such that for all $\overline{c} \in \mathbb{N}_+^k$: $\mathcal{B}[p(\overline{x})]$ has exactly $p(\overline{c})$ accepting runs on input $a^{\overline{c}}$.*

Proof. The lemma is shown by induction on the construction of the polynomial $p(\overline{x})$. Büchi automata for the polynomials $p(\overline{x}) = 1$ and $p(\overline{x}) = x_i$ are easily build. Now let $\mathcal{B}[p_1(\overline{x})]$ and $\mathcal{B}[p_2(\overline{x})]$ be already constructed. Then it is easily seen that the disjoint union of these two Büchi-automata can serve has $\mathcal{B}[p_1(\overline{x}) + p_2(\overline{x})]$. The construction of the Büchi-automaton $\mathcal{B}[p_1(\overline{x}) \cdot p_2(\overline{x})]$ uses Choueka's flag construction (cf. [6, 30, 26]):

Let $\mathcal{B}[p_i(\overline{x})] = (Q_i, \Gamma, I_i, \Delta_i, F_i)$ for $i \in \{1, 2\}$ and set

$$\mathcal{B}[p_1(\overline{x}) \cdot p_2(\overline{x})] = (Q_1 \times Q_2 \times \{1, 2\}, \Gamma, I_1 \times I_2 \times \{1\}, \Delta, F_1 \times Q_2 \times \{1\}),$$

where $((p_1, p_2, m), a, (q_1, q_2, n)) \in \Delta$ if and only if

- $(p_1, a, q_1) \in \Delta_1$ and $(p_2, a, q_2) \in \Delta_2$, and
- if $p_m \notin F_m$ then $n = m$ and if $p_m \in F_m$ then $n = 3 - m$.

Hence the runs of $\mathcal{B}[p_1(\overline{x}) \cdot p_2(\overline{x})]$ on the ω-word $a^{\overline{c}}$ consist of a run of $\mathcal{B}[p_1(\overline{x})]$ and of $\mathcal{B}[p_2(\overline{x})]$ on $a^{\overline{c}}$. The "flag" $m \in \{1, 2\}$ in (p_1, p_2, m) signals that the automaton waits for an accepting state of $\mathcal{B}[p_m(\overline{x})]$. As soon as such an accepting state is seen, the flag toggles its value. Hence accepting runs of $\mathcal{B}[p_1(\overline{x}) \cdot p_2(\overline{x})]$ correspond to pairs of accepting runs of $\mathcal{B}[p_1(\overline{x})]$ and of $\mathcal{B}[p_2(\overline{x})]$. Therefore, the number of accepting runs of $\mathcal{B}[p_1(\overline{x}) \cdot p_2(\overline{x})]$ on $a^{\overline{c}}$ equals the product of the numbers of accepting runs of $\mathcal{B}[p_1(\overline{x})]$ and of $\mathcal{B}[p_2(\overline{x})]$ on $a^{\overline{c}}$. □

Lemma 20. *From a given boolean combination $\psi(x_1, \ldots, x_m, X_1, \ldots, X_n)$ of set constraints on set variables $X_1, \ldots, X_n$ and individual variables $x_1, \ldots, x_m$ one can construct effectively a deterministic Büchi automaton $\mathcal{A}_\psi$ over the alphabet $\{0, 1\}^n \times \{a, \diamond\}^m$ such that for all $X_1, \ldots, X_n \subseteq \mathbb{N}_+, \overline{c} \in \mathbb{N}_+^m$, the following holds:*

$$w_{X_1} \otimes \cdots \otimes w_{X_n} \otimes a^{\overline{c}} \in L(\mathcal{A}_\psi) \iff \psi(\overline{c}, X_1, \ldots, X_n) \text{ holds.}$$

Proof. Since set constraints are closed under negation, we can assume that ψ is a positive boolean combination. Then the claim is trivial for a single set constraint. Since ω-languages accepted by deterministic Büchi automata are effectively closed under intersection and union, the result follows. □

In the next lemma, k, ℓ, n, p_i, and ψ_i ($1 \leq i \leq \ell$) are taken from the definition of our Π_n^1-set A in (8).

Lemma 21. *From $1 \leq i \leq \ell$, one can construct a Büchi automaton $\mathcal{A}_i$ with the following property: For all $\overline{X} \in (2^{\mathbb{N}_+})^n$, $\overline{z} \in \mathbb{N}_+^k$, and $x, y, z_{k+1} \in \mathbb{N}_+$, the number of accepting runs of $\mathcal{A}_i$ on the word $w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$ equals*

$$\begin{cases} C(1,2) & \text{if } \psi_i(x, y, \overline{z}, \overline{X}) \text{ holds} \\ C(p_i(x, y, \overline{z}) + z_{k+1}, q_i(x, y, \overline{z}) + z_{k+1}) & \text{otherwise.} \end{cases}$$

Proof. By Lemma 19, one can construct a Büchi automaton $\mathcal{B}_i$, which has precisely $C(p_i(x,y,\overline{z})+z_{k+1}, q_i(x,y,\overline{z})+z_{k+1})$ many accepting runs on the ω-word $w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$. Secondly, one builds deterministic Büchi automata $\mathcal{C}_i$ and $\overline{\mathcal{C}}_i$ accepting a word $w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$ if and only if the disjunction $\psi_i(x,y,\overline{z},\overline{X})$ of set constraints is satisfied (not satisfied, resp.) which is possible by Lemma 20.

Let $\mathcal{A}$ be the result of applying the flag construction to $\overline{\mathcal{C}}_i$ and $\mathcal{B}_i$, and let $\overline{X} \in (2^{\mathbb{N}_+})^n$, $\overline{z} \in \mathbb{N}_+^k$, and $x, y, z_{k+1} \in \mathbb{N}_+$. Since $\overline{\mathcal{C}}_i$ is deterministic, the number of accepting runs of $\overline{\mathcal{C}}_i$ on the word $w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$ is either 0 (if $\psi_i(x,y,\overline{z},\overline{X})$ holds) or 1 (if $\psi_i(x,y,\overline{z},\overline{X})$ does not hold). Since the flag construction multiplies the number of accepting runs of the two automata, it follows that the number of accepting runs of $\mathcal{A}$ on the word $w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$ is

$$\begin{cases} 0 & \text{if } \psi_i(x,y,\overline{z},\overline{X}) \text{ holds} \\ C(p_i(x,y,\overline{z})+z_{k+1}, q_i(x,y,\overline{z})+z_{k+1}) & \text{otherwise.} \end{cases}$$

Hence the disjoint union of $\mathcal{A}$ and $C(1,2)$ many copies of $\mathcal{C}_i$ has the desired properties. □

Lemma 22. *One can construct an injectively ω-automatic forest $\mathcal{H}' = (L', E')$ of height 1 such that*

- *the set of roots equals* $\{1,\ldots,\ell\} \otimes (\otimes_n(\{0,1\}^\omega)) \otimes (\otimes_{k+3}(a^+)) \cup (b^+ \otimes b^+)$,
- *for* $1 \le i \le \ell$, $\overline{X} \in (2^{\mathbb{N}_+})^n$, $x, y, z_{k+1} \in \mathbb{N}_+$ *and* $\overline{z} \in \mathbb{N}_+^k$, *we have*

$$\mathcal{H}'(i \otimes w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}) \cong T'[\overline{X}, x, y, \overline{z}, z_{k+1}, i] \text{ and}$$

- *for* $e_1, e_2 \in \mathbb{N}_+$, *we have*
$$\mathcal{H}'(b^{(e_1,e_2)}) \cong S[e_1,e_2]\,.$$

Proof. Using Lemma 19 (with the polynomial $p = C(x_1,x_2)$) and Lemma 21, we can construct a Büchi automaton $\mathcal{A}$ accepting $\{1,\ldots,\ell\} \otimes (\otimes_n(\{0,1\}^\omega)) \otimes (\otimes_{k+3}(a^+)) \cup (b^+ \otimes b^+)$ such that the number of accepting runs of $\mathcal{A}$ on the ω-word u equals

(i) $C(e_1,e_2)$ if $u = b^{(e_1,e_2)}$,

(ii) $C(1,2)$ if $u = i \otimes w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$ such that $\psi_i(x,y,\overline{z},\overline{X})$ holds, and

(iii) $C(p_i(x,y,\overline{z})+z_{k+1}, q_i(x,y,\overline{z})+z_{k+1})$ if $u = i \otimes w_{\overline{X}} \otimes a^{(x,y,\overline{z},z_{k+1})}$ such that $\psi_i(x,y,\overline{z},\overline{X})$ does not hold.

Let $\mathrm{Run}_{\mathcal{A}}$ denote the set of accepting runs of $\mathcal{A}$. Note that this is a regular ω-language over the alphabet Δ of transitions of $\mathcal{A}$. Now the forest $\mathcal{H}'$ is defined as follows:

- Its universe equals $L(\mathcal{A}) \cup \mathrm{Run}_{\mathcal{A}}$.
- There is an edge (u,v) if and only if $v \in \mathrm{Run}_{\mathcal{A}}$ is a accepting run of $\mathcal{A}$ on $u \in L(\mathcal{A})$.

It is clear that $\mathcal{H}'$ is an injectively ω-automatic forest of height 1 with set of roots $L(\mathcal{A})$ as required. Note that (i)-(iii) describe the number of leaves of the height-1 tree rooted at $u \in L(\mathcal{A})$. By (i), we therefore get immediately $\mathcal{H}'(b^{(e_1,e_2)}) \cong S[e_1, e_2]$. Comparing the numbers in (ii) and (iii) with the definition of the tree $T'[\overline{X}, x, y, \overline{z}, z_{k+1}, i]$ in (11) completes the proof. □

From $\mathcal{H}' = (L', E')$, we build an injectively ω-automatic dag $\mathcal{D}$ as follows:

- The domain of $\mathcal{D}$ is the set $(\otimes_n(\{0,1\}^\omega) \otimes a^+ \otimes a^+) \ \cup \ b^* \ \cup \ \big(\$^* \otimes L'\big)$.
- For $u, v \in L'$, the words $\$^i \otimes u$ and $\$^j \otimes v$ are connected if and only if $i = j$ and $(u, v) \in E'$. In other words, the restriction of $\mathcal{D}$ to $\$^* \otimes L'$ is isomorphic to $\mathcal{H}'^{\aleph_0}$.
- For all $\overline{X} \in (2^{\mathbb{N}_+})^n$, $x, y \in \mathbb{N}_+$, the new root $w_{\overline{X}} \otimes a^{(x,y)}$ is connected to all nodes in
$$\$^* \otimes \big((\{1, \ldots, \ell\} \otimes w_{\overline{X}} \otimes a^{(x,y)} \otimes (\otimes_{k+1}(a^+))) \cup \{b^{(e_1,e_2)} \mid e_1 \neq e_2\}\big)\ .$$
- The new root ε is connected to all nodes in $\$^* \otimes \{b^{(e_1,e_2)} \mid e_1 \neq e_2\}$.
- For all $m \in \mathbb{N}_+$, the new root b^m is connected to all nodes in
$$\$^* \otimes \{b^{(e_1,e_2)} \mid e_1 \neq e_2 \vee e_1 = e_2 \geq m\}.$$

It is easily seen that $\mathcal{D}$ is an injectively ω-automatic dag. Let $\mathcal{H}'' = \mathrm{unfold}(\mathcal{D})$ which is also injectively ω-automatic by Lemma 18. Then, for all $\overline{X} \in (2^{\mathbb{N}_+})^n$, $x, y, m \in \mathbb{N}_+$, we have (L22 refers to Lemma 22):

$$\begin{aligned}
\mathcal{H}''(w_{\overline{X}} \otimes a^{(x,y)}) &\cong (w_{\overline{X}} \otimes a^{(x,y)}) \circ \begin{pmatrix} \biguplus\{\mathcal{H}'(i \otimes w_{\overline{X}} \otimes a^{(x,y,\overline{z})}) \mid 1 \leq i \leq \ell, \overline{z} \in \mathbb{N}_+^{k+1}\} \uplus \\ \biguplus\{\mathcal{H}'(b^{(e_1,e_2)}) \mid e_1 \neq e_2\} \end{pmatrix}^{\aleph_0} \\
&\overset{\text{L22}}{\cong} r \circ \begin{pmatrix} \biguplus\{T'[\overline{X}, x, y, \overline{z}, i] \mid \overline{z} \in \mathbb{N}_+^{k+1}, 1 \leq i \leq \ell\} \uplus \\ \biguplus\{S[e_1, e_2] \mid e_1 \neq e_2\} \end{pmatrix}^{\aleph_0} \\
&\overset{(12)}{=} T''[\overline{X}, x, y]
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}''(\varepsilon) &\cong \varepsilon \circ \Big(\biguplus\{\mathcal{H}'(b^{(e_1,e_2)}) \mid e_1 \neq e_2\}\Big)^{\aleph_0} \\
&\overset{\text{L22}}{\cong} r \circ \Big(\biguplus\{S[e_1, e_2] \mid e_1 \neq e_2\}\Big)^{\aleph_0} \\
&\overset{(13)}{=} U''[\omega]
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}''(b^m) &\cong b^m \circ \Big(\biguplus\{\mathcal{H}'(b^{(e_1,e_2)}) \mid e_1 \neq e_2 \vee e_1 = e_2 \geq m\}\Big)^{\aleph_0} \\
&\overset{\text{L22}}{\cong} r \circ \Big(\biguplus\{S[e_1, e_2] \mid e_1 \neq e_2 \vee e_1 = e_2 \geq m\}\Big)^{\aleph_0} \\
&\overset{(13)}{=} U''[m]
\end{aligned}$$

From $\mathcal{H}'' = (L'', E'')$ we build an injectively ω-automatic dag $\mathcal{D}_0$ as follows:

- The domain of $\mathcal{D}_0$ is the set $(\otimes_n(\{0,1\}^\omega) \otimes a^+) \cup \{\varepsilon, b\} \cup (\$^* \otimes L'')$.
- For $u, v \in L''$, the words $\$^i \otimes u$ and $\$^j \otimes v$ are connected by an edge if and only if $i = j$ and $(u, v) \in E''$, i.e., the restriction of $\mathcal{D}_0$ to $\$^* \otimes L''$ is isomorphic to $\mathcal{H}''^{\aleph_0}$.
- For $\overline{X} \in (2^{\mathbb{N}_+})^n$, $x \in \mathbb{N}_+$ we connect the new root $w_{\overline{X}} \otimes a^x$ to all nodes in

$$\$^* \otimes \left((w_{\overline{X}} \otimes a^x \otimes a^+) \;\cup\; b^+\right) \subseteq \$^* \otimes L''.$$

- We connect the new root ε to all nodes in $\$^* \otimes b^*$.
- We connect the new root b to all nodes in $\$^* \otimes b^+$.

Then $\mathcal{D}_0$ is an injectively ω-automatic dag of height 3 and we set $\mathcal{H}_0 = \text{unfold}(\mathcal{D}_0)$. We have the following:

- The set of roots of $\mathcal{H}_0$ is $(\otimes_n(\{0,1\}^\omega) \otimes a^+) \cup \{\varepsilon, b\}$.
- For all $\overline{X} \in (2^{\mathbb{N}_+})^n$, $x \in \mathbb{N}_+$ we have:

$$\begin{aligned}
\mathcal{H}_0(w_{\overline{X}} \otimes a^x) &\cong (w_{\overline{X}} \otimes a^x) \circ \begin{pmatrix} \biguplus\{\mathcal{H}''(b^m) \mid m \in \mathbb{N}_+\} \uplus \\ \biguplus\{\mathcal{H}''(w_{\overline{X}} \otimes a^x \otimes a^y) \mid y \in \mathbb{N}_+\} \end{pmatrix}^{2^{\aleph_0}} \\
&\cong r \circ \left(\biguplus\{U''[m] \mid m \in \mathbb{N}_+\} \uplus \biguplus\{T''[\overline{X}, x, y] \mid y \in \mathbb{N}_+\}\right)^{\aleph_0} \\
&\overset{(14)}{=} T_0[\overline{X}, x] \\
\mathcal{H}_0(\varepsilon) &\cong \varepsilon \circ \left(\biguplus\{\mathcal{H}''(b^m) \mid m \in \mathbb{N}\}\right)^{\aleph_0} \\
&\cong r \circ \left(\biguplus\{U''[\kappa] \mid \kappa \in \mathbb{N}_+ \cup \{\omega\}\}\right)^{\aleph_0} \\
&\overset{(15)}{=} U_0[0] \\
\mathcal{H}_0(b) &\cong b \circ \left(\biguplus\{\mathcal{H}''(b^m) \mid m \in \mathbb{N}_+\}\right)^{\aleph_0} \\
&\cong r \circ \left(\biguplus\{U''[m] \mid m \in \mathbb{N}_+\}\right)^{\aleph_0} \\
&\overset{(16)}{=} U_0[1]
\end{aligned}$$

These identities settle the induction base for the proof of Lemma 17.

We now construct the forests $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \ldots, \mathcal{H}_n$ inductively. For $0 \le m < n$, suppose we have obtained an injectively ω-automatic forest $\mathcal{H}_m = (L_m, E_m)$ as described in Lemma 17. The forest $\mathcal{H}_{m+1}$ is constructed as follows, where $\alpha = (m \bmod 2) \in \{0, 1\}$:

- The domain of $\mathcal{H}_{m+1}$ is $(\otimes_{n-m-1}(\{0,1\}^\omega) \otimes a^+) \cup \{\varepsilon, b\} \cup (\{\$_1, \$_2\}^\omega \otimes L_m)$.
- For $u, v \in L_m$ and $u', v' \in \{\$_1, \$_2\}^\omega$, the words $u' \otimes u$ and $v' \otimes v$ are connected by an edge if and only if $u' = v'$ and $(u, v) \in E_m$, i.e., the restriction of $\mathcal{D}_{m+1}$ to $\{\$_1, \$_2\}^\omega \otimes L_m$ is isomorphic to $\mathcal{H}_m^{2^{\aleph_0}}$.
- For all $\overline{X} \in (2^{\mathbb{N}_+})^{n-m-1}$ and all $x \in \mathbb{N}_+$, connect the new root $w_{\overline{X}} \otimes a^x$ to all nodes from
$$\{\$_1, \$_2\}^\omega \otimes \left(w_{\overline{X}} \otimes \{0,1\}^\omega \otimes a^x \cup b^\alpha\right).$$
- Connect the new root ε to all nodes from $\{\$_1, \$_2\}^\omega \otimes \{\varepsilon, b^\alpha\}$.
- Connect the new root b to all nodes from $\{\$_1, \$_2\}^\omega \otimes \{b, b^\alpha\}$.

In this way we obtain the injectively ω-automatic forest $\mathcal{H}_{m+1}$ such that:

- The set of roots of $\mathcal{H}_{m+1}$ is $(\otimes_{n-m-1}(\{0,1\}^\omega) \otimes a^+) \cup \{\varepsilon, b\}$.
- For $\overline{X} \in (2^{\mathbb{N}_+})^{n-m-1}$ and $x \in \mathbb{N}_+$ we have (IH stands for induction hypothesis):

$$\begin{aligned}
\mathcal{H}_{m+1}(w_{\overline{X}} \otimes a^x) &\cong (w_{\overline{X}} \otimes a^x) \circ \begin{pmatrix} \biguplus\{\mathcal{H}_m(w_{\overline{X}} \otimes w_{X_{n-m}} \otimes a^x) \mid X_{n-m} \subseteq \mathbb{N}_+\} \uplus \\ \mathcal{H}_m(b^\alpha) \end{pmatrix}^{2^{\aleph_0}} \\
&\overset{\text{IH}}{\cong} r \circ \left(\biguplus\{T_m[\overline{X}, X_{n-m}, x] \mid X_{n-m} \subseteq \mathbb{N}_+\} \uplus U_m[\alpha]\right)^{2^{\aleph_0}} \\
&\overset{(17)}{\cong} T_{m+1}[\overline{X}, x] \\
\mathcal{H}_{m+1}(\varepsilon) &\cong \varepsilon \circ (\mathcal{H}_m(\varepsilon) \uplus \mathcal{H}_m(b^\alpha))^{2^{\aleph_0}} \\
&\overset{\text{IH}}{\cong} r \circ (U_m[0] \uplus U_m[\alpha])^{2^{\aleph_0}} \\
&\overset{(18)}{\cong} U_{m+1}[0] \\
\mathcal{H}_{m+1}(b) &\cong b \circ (\mathcal{H}_m(b) \uplus \mathcal{H}_m(b^\alpha))^{2^{\aleph_0}} \\
&\overset{\text{IH}}{\cong} r \circ (U_m[1] \uplus U_m[\alpha])^{2^{\aleph_0}} \\
&\overset{(18)}{\cong} U_{m+1}[1]
\end{aligned}$$

This concludes the proof of Lemma 17 and hence of Proposition 10. Consequently, the main results (Corollaries 11 and 12) of this section hold.

6. ω-automatic trees of height 3

Recall that the isomorphism problem $\mathsf{Iso}(\mathcal{T}_2^{\mathrm{i}})$ is arithmetical by Theorem 8 and that $\mathsf{Iso}(\mathcal{T}_4^{\mathrm{i}})$ is not by Corollary 12. In this section, we modify the proof of Proposition 10 in order to show that already $\mathsf{Iso}(\mathcal{T}_3^{\mathrm{i}})$ is not arithmetical:

Theorem 23. *There exists a tree U such that $\{P \in \mathcal{T}_3^{\mathrm{i}} \mid \mathcal{S}(P) \cong U\}$ is Π_1^1-hard. Hence the isomorphism problem $\mathsf{Iso}(\mathcal{T}_3^{\mathrm{i}})$ for injectively ω-automatic trees of height 3 is Π_1^1-hard.*

So let $A \subseteq \mathbb{N}_+$ be some set from Π_1^1. By Proposition 9 it can be written as

$$A = \{x \in \mathbb{N}_+ : \forall X\ \exists y\ \forall \overline{z} : \bigwedge_{i=1}^{\ell} p_i(x,y,\overline{z}) \neq q_i(x,y,\overline{z}) \vee \psi_i(x,y,\overline{z},X)\},$$

where p_i and q_i are polynomials with coefficients in $\mathbb{N}$ and ψ_i is a disjunction of set constraints. As in Section 5, let $\varphi_{-1}(x,y,X)$ denote the subformula starting with $\forall \overline{z}$, and let $\varphi_0(x,X) = \exists y : \varphi_{-1}(x,y,X)$. We reuse the trees $T'[X,x,y,\overline{z},z_{k+1},i]$ of height 1 defined in (11). Recall that they are all of the form $S[e_1,e_2]$ and therefore have an even number of leaves (since the range of the polynomial $C : \mathbb{N}_+^2 \to \mathbb{N}_+$ from (10) consists of even numbers). For $e \in \mathbb{N}_+$, let $S[e]$ denote the height-1 tree with $2e+1$ leaves.

Recall that the tree $T''[X,x,y]$ from (12) encodes the set

$$\{(e_1,e_2) \mid e_1 \neq e_2\} \cup \{(p_i(x,y,\overline{z})+z_{k+1}, q_i(x,y,\overline{z})+z_{k+1}) \mid 1 \leq i \leq \ell, z_{k+1} \in \mathbb{N}_+, \overline{z} \in \mathbb{N}_+^k, \psi_i(x,y,\overline{z},\overline{X}) \text{ does not hold}\}\,.$$

We now modify the construction of this tree such that, in addition, it also encodes the set $X \subseteq \mathbb{N}_+$:

$$\widehat{T}[X,x,y] = r \circ \begin{pmatrix} \biguplus\{S[e] \mid e \in X\} \uplus \biguplus\{S[e_1,e_2] \mid e_1 \neq e_2\} \uplus \\ \biguplus\{T'[\overline{X},x,y,\overline{z},z_{k+1}i] \mid \overline{z} \in \mathbb{N}_+^k, z_{k+1} \in \mathbb{N}_+, 1 \leq i \leq \ell\} \end{pmatrix}^{\aleph_0}$$

In a similar spirit, we define $\widehat{U}[\kappa,X]$ for $X \subseteq \mathbb{N}_+$ and $\kappa \in \mathbb{N}_+ \cup \{\omega\}$:

$$\widehat{U}[\kappa,X] = r \circ \begin{pmatrix} \biguplus\{S[e] \mid e \in X\} \uplus \biguplus\{S[e_1,e_2] \mid e_1 \neq e_2\} \uplus \\ \biguplus\{S[e,e] \mid \kappa \leq e < \omega\} \end{pmatrix}^{\aleph_0}$$

Then $\widehat{T}[X,x,y] \cong \widehat{U}[\omega,Y]$ if and only if $X = Y$ and $T''[X,x,y] \cong U''[\omega]$, i.e., if and only if $X = Y$ and $\varphi_{-1}(x,y,X)$ holds by Lemma 13(b). Finally, we set

$$T[x] = r \circ \left(\biguplus\{\widehat{U}[\kappa,X] \mid X \subseteq \mathbb{N}_+, \kappa \in \mathbb{N}_+\} \uplus \biguplus\{\widehat{T}[X,x,y] \mid X \subseteq \mathbb{N}_+, y \in \mathbb{N}_+\}\right)^{\aleph_0},$$
$$U = r \circ \left(\biguplus\{\widehat{U}[\kappa,X] \mid X \subseteq \mathbb{N}_+, \kappa \in \mathbb{N}_+ \cup \{\omega\}\}\right)^{\aleph_0}.$$

Lemma 24. *Let $x \in \mathbb{N}_+$. Then $T[x] \cong U$ if and only if $x \in A$.*

Proof. Suppose $x \in A$. To prove $T[x] \cong U$, it suffices to show that any height-2 subtree of $T[x]$ is a subtree of U and vice versa. First, let $X \subseteq \mathbb{N}_+$ and $y \in \mathbb{N}_+$. Then, by Lemma 13(a), there exists $\kappa \in \mathbb{N}_+ \cup \{\omega\}$ with $T''[X,x,y] \cong U''[\kappa]$ and therefore $\widehat{T}[X,x,y] \cong$

$\widehat{U}[\kappa, X]$, i.e., $\widehat{T}[X, x, y]$ appears in U. Secondly, let $X \subseteq \mathbb{N}_+$. From $x \in A$, we can infer that there exists some $y \in \mathbb{N}_+$ with $\varphi_{-1}(x, y, X)$. Then Lemma 13(b) implies $U''[\omega] \cong T''[X, x, y]$ and therefore $\widehat{U}[\omega, X] \cong \widehat{T}[X, x, y]$, i.e., $\widehat{U}[\omega, X]$ appears in $T[x]$. Thus, any height-2 subtree of $T[x]$ is a subtree of U and vice versa.

Conversely suppose $T[x] \cong U$. Let $X \subseteq \mathbb{N}_+$. Then $\widehat{U}[\omega, X]$ appears in U and therefore in $T[x]$. Since $\widehat{U}[\omega, X] \ncong \widehat{U}[\kappa, Y]$ for all $\kappa \in \mathbb{N}_+$ and $Y \subseteq \mathbb{N}_+$, there exists some $y \in \mathbb{N}_+$ with $\widehat{U}[\omega, X] \cong \widehat{T}[X, x, y]$. Thus, we have $T''[X, x, y] \cong U''[\omega]$. Lemma 13(b) implies that $\varphi_{-1}(x, y, X)$ holds. We have shown that $x \in A$. □

6.1. Injective ω-automaticity

We follow closely the construction for $m = 0$ from Section 5.2.

Lemma 25. *There exists an injectively ω-automatic forest $\mathcal{H}' = (L', E')$ of height 1 such that:*

- *The set of roots equals $\{1, \ldots, \ell\} \otimes \{0,1\}^\omega \otimes (\otimes_{k+3}(a^+)) \cup (b^+ \otimes b^+) \cup c^+$.*
- *For $1 \leq i \leq \ell$, $X \subseteq \mathbb{N}_+$, $x, y, z_{k+1} \in \mathbb{N}_+$ and $\overline{z} \in \mathbb{N}_+^k$, we have*
$$\mathcal{H}'(i \otimes w_X \otimes a^{(x,y,\overline{z},z_{k+1})}) \cong T'[X, x, y, \overline{z}, z_{k+1}, i].$$
- *For $e_1, e_2 \in \mathbb{N}_+$, we have*
$$\mathcal{H}'(b^{(e_1,e_2)}) \cong S[e_1, e_2].$$
- *For $e \in \mathbb{N}_+$, we have $\mathcal{H}'(c^e) \cong S[e]$.*

Proof. Using Lemma 19 twice (with the polynomial $C(x_1, x_2)$ and with the polynomial $2x_1 + 1$) and Lemma 21, we can construct a Büchi automaton $\mathcal{A}$ accepting $\{1, \ldots, \ell\} \otimes \{0,1\}^\omega \otimes (\otimes_{k+3}(a^+)) \cup (b^+ \otimes b^+) \cup c^+$ such that the number of accepting runs of $\mathcal{A}$ on the ω-word u equals

(i) $C(e_1, e_2)$ if $u = b^{(e_1,e_2)}$,

(ii) $2e + 1$ if $u = c^e$,

(iii) $C(1, 2)$ if $u = i \otimes w_X \otimes a^{(x,y,\overline{z},z_{k+1})}$ such that $\psi_i(x, y, \overline{z}, X)$ holds, and

(iv) $C(p_i(x, y, \overline{z}) + z_{k+1}, q_i(x, y, \overline{z}) + z_{k+1})$ if $u = i \otimes w_X \otimes a^{(x,y,\overline{z},z_{k+1})}$ such that $\psi_i(x, y, \overline{z}, X)$ does not hold.

The rest of the proof is the same as that of Lemma 22. □

From $\mathcal{H}' = (L', E')$, we build an injectively ω-automatic dag $\mathcal{D}$ as follows:

- The domain of $\mathcal{D}$ is the set $(\{0,1\}^\omega \otimes a^+ \otimes a^+) \cup (\{0,1\}^\omega \otimes b^*) \cup (\$^* \otimes L')$.

- For $u, v \in L'$, the words $\$^i \otimes u$ and $\$^j \otimes v$ are connected if and only if $i = j$ and $(u, v) \in E'$. In other words, the restriction of $\mathcal{D}$ to $\$^* \otimes L'$ is isomorphic to $\mathcal{H}'^{\aleph_0}$.
- For all $X \subseteq \mathbb{N}_+$, $x, y \in \mathbb{N}_+$, the new root $w_X \otimes a^{(x,y)}$ is connected to all nodes in
$$\$^* \otimes \big((\{1, \ldots, \ell\} \otimes w_X \otimes a^{(x,y)} \otimes (\otimes_{k+1}(a^+))) \cup \{b^{(e_1,e_2)} \mid e_1 \neq e_2\} \cup \{c^e \mid e \in X\}\big)\ .$$
- For all $X \subseteq \mathbb{N}_+$, the new root $w_X \otimes \varepsilon$ is connected to all nodes in
$$\$^* \otimes (\{b^{(e_1,e_2)} \mid e_1 \neq e_2\} \cup \{c^e \mid e \in X\}).$$
- For all $X \subseteq \mathbb{N}_+$ and $m \in \mathbb{N}_+$, the new root $w_X \otimes b^m$ is connected to all nodes in
$$\$^* \otimes (\{b^{(e_1,e_2)} \mid e_1 \neq e_2 \vee e_1 = e_2 \geq m\} \cup \{c^e \mid e \in X\}).$$

It is easily seen that $\mathcal{D}$ is an injectively ω-automatic dag. Let $\mathcal{H}'' = \text{unfold}(\mathcal{D})$ which is also injectively ω-automatic by Lemma 18. Now computations analogous to those on page 19 (using Lemma 25 instead of Lemma 22) yield for all $X \subseteq \mathbb{N}_+$ and $x, y, m \in \mathbb{N}_+$:

$$\begin{aligned}\mathcal{H}''(w_X \otimes a^{(x,y)}) &\cong \widehat{T}[X, x, y] \\ \mathcal{H}''(w_X \otimes \varepsilon) &\cong \widehat{U}[\omega, X] \\ \mathcal{H}''(w_X \otimes b^m) &\cong \widehat{U}[m, X]\end{aligned}$$

From $\mathcal{H}'' = (L'', E'')$, we build an injectively ω-automatic dag $\mathcal{D}_0$ as follows:

- The domain of $\mathcal{D}_0$ equals $a^* \ \cup\ \$^* \otimes L''$.
- For $u, v \in L''$, the words $\$^i \otimes u$ and $\$^j \otimes v$ are connected by an edge if and only if $i = j$ and $(u, v) \in E''$. Hence the restriction of $\mathcal{D}_0$ to $\$^* \otimes L''$ is isomorphic to $\mathcal{H}''^{\aleph_0}$.
- For $x \in \mathbb{N}_+$, the new root a^x is connected to all nodes in
$$\$^* \otimes \big(\{0,1\}^\omega \otimes b^+ \ \cup\ \{0,1\}^\omega \otimes a^x \otimes a^+\big)\,.$$
- The new root ε is connected to all nodes in $\$^* \otimes \{0,1\}^\omega \otimes b^*$.

Then $\mathcal{D}_0$ is an injectively ω-automatic dag of height 3 and we set $\mathcal{H}_0 = \text{unfold}(\mathcal{D}_0)$. The set of roots of $\mathcal{H}_0$ is a^*. Calculations similar to those on page 22 then yield $\mathcal{H}_0(\varepsilon) \cong U$ and $\mathcal{H}_0(a^x) \cong T[x]$ for $x \in \mathbb{N}_+$. Hence, $T[x]$ is (effectively) an injectively ω-automatic tree. Now Lemma 24 finishes the proof of the first statement of Theorem 23, the second follows immediately.

Remark 26. *In our previous paper [21], we proved that the isomorphism problem for* automatic trees *of height $n \geq 2$ is hard (in fact complete) for level Π^0_{2n-3} of the arithmetical hierarchy. For this construction we used the fact that Π^0_{2n+1}-sets can be defined by the quantifier prefix $\exists^\infty x_1 \cdots \exists^\infty x_n \forall y$, see [27, Theorem XVIII] (in our construction, a single $\exists^\infty$-quantifier increases the height of the trees only by one). An analogous characterization for Π^1_{2n+1}-sets clearly fails.*

7. Upper bounds assuming CH

In the following, we will identify an ω-word $w \in \Gamma^\omega$ with the function $w : \mathbb{N}_+ \to \Gamma$, (and hence with a second-order object) where $w(i) = w[i]$. We need the following lemma:

Lemma 27. *From a given Büchi automaton M over an alphabet Γ one can construct an arithmetical predicate $\mathsf{acc}_M(u)$ (where $u : \mathbb{N}_+ \to \Gamma$) such that: $u \in L(M)$ if and only if $\mathsf{acc}_M(u)$ holds.*

Proof. First, let M be a *deterministic* Büchi-automaton with set of states Q. For a given ω-word $u\colon \mathbb{N}_+ \to \Gamma$ and $i \in \mathbb{N}$ let $q(u,i) \in Q$ be the unique state that is reached by M after reading the length-i prefix of u. Note that $q(u,i)$ is computable from i (if u is given as an oracle), hence $q(u,i)$ is arithmetically definable and u is accepted by M iff

$$\bigvee_{f \in F} \forall x \in \mathbb{N}_+ \exists y \geq x : q(u,y) = f\,.$$

Finally note that every regular ω-language is (effectively) a Boolean combination of ω-languages accepted by deterministic Büchi-automata (cf. [26, Theorem II.9.3]). □

Theorem 28. *Assuming* **CH**, *the isomorphism problem $\mathsf{Iso}(\mathcal{T}_n)$ belongs to Π^1_{2n-4} for $n \geq 3$.*

Proof. Consider trees $T_i = \mathcal{S}(P_i)$ for $P_1, P_2 \in \mathcal{T}_n$. Define the forest $F = (V, \leq)$ as $F = T_1 \uplus T_2$. Let E be the edge relation of F. Recall that $E(v)$ denotes the set of children of $v \in V$. Let us fix an ω-automatic presentation $P = (\Sigma, M, M_\equiv, M_E)$ for the graph (V, E). In the following, for $u \in L(M)$ we write $F(u)$ for the subtree $F([u]_{R(M_\equiv)})$ rooted in the F-node $[u]_{R(M_\equiv)}$ represented by the ω-word u. Similarly, we write $E(u)$ for $E([u]_{R(M_\equiv)})$. We will define a $\Pi^1_{2n-2k-4}$-predicate $\mathsf{iso}_k(u_1, u_2)$, where $u_1, u_2 \in L(M)$ are on level k in F. This predicate expresses that $F(u_1) \cong F(u_2)$.

As induction base, let $k = n-2$. Then the trees $F(u_1)$ and $F(u_2)$ have height at most 2. Then, as in the proof of Theorem 8, we have $F(u_1) \cong F(u_2)$ if and only if the following holds for all $\kappa, \lambda \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$:

$$F \models \Big(\exists^\kappa x \in V : (([u_1], x) \in E \wedge \exists^\lambda y \in V : (x,y) \in E)\Big) \leftrightarrow \Big(\exists^\kappa x \in V : (([u_2], x) \in E \wedge \exists^\lambda y \in V : (x,y) \in E)\Big)\,.$$

Note that by Theorem 2, one can compute from $\kappa, \lambda \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ a Büchi automaton $M_{\kappa,\lambda}$ accepting the set of convolutions of pairs of ω-words (u_1, u_2) satisfying the above formula. Hence $F(u_1) \cong F(u_2)$ if and only if the following arithmetical predicate holds:

$$\forall \kappa, \lambda \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\} : \mathsf{acc}_{M_{\kappa,\lambda}}(u_1 \otimes u_2)\,.$$

Now let $0 \leq k < n-2$. We first introduce a few notations. For a set A, let $\mathsf{count}(A)$ denote the set of all countable (possibly finite) subsets of A. For $\kappa \in \mathbb{N} \cup \{\aleph_0\}$ we denote with $[\kappa]$

the set $\{0, \ldots, \kappa - 1\}$ (resp. $\mathbb{N}$) in case $\kappa \in \mathbb{N}$ ($\kappa = \aleph_0$). For a function $f : (A \times B) \to C$ and $a \in A$ let $f[a] : B \to C$ denote the function with $f[a](b) = f(a, b)$.

On an abstract level, the formula $\mathsf{iso}_k(u_1, u_2)$ is

$$\big(\forall x \in E(u_1)\ \exists y \in E(u_2) : \mathsf{iso}_{k+1}(x, y)\big)\ \wedge \tag{19}$$

$$\big(\forall x \in E(u_2)\ \exists y \in E(u_1) : \mathsf{iso}_{k+1}(x, y)\big)\ \wedge \tag{20}$$

$$\forall X_1 \in \mathsf{count}(E(u_1))\, \forall X_2 \in \mathsf{count}(E(u_2)) : \tag{21}$$

$$\exists x, y \in X_1 \cup X_2 : \neg \mathsf{iso}_{k+1}(x, y) \vee \tag{22}$$

$$\exists x \in X_1 \cup X_2\ \exists y \in (E(u_1) \cup E(u_2)) \setminus (X_1 \cup X_2) : \mathsf{iso}_{k+1}(x, y) \vee \tag{23}$$

$$|X_1| = |X_2|\,. \tag{24}$$

Line (19) and (20) express that the children of u_1 and u_2 realize the same isomorphism types of trees of height $\leq n - k - 1$. The rest of the formula expresses that if a certain isomorphism type τ of height-$(n - k - 1)$ trees appears countably many times below u_1 then it appears with the same multiplicity below u_2 and vice versa. Assuming **CH** and the correctness of iso_{k+1}, the formula $\mathsf{iso}_k(u_1, u_2)$ expresses indeed that $F(u_1) \cong F(u_2)$.

In the above definition of $\mathsf{iso}_k(u_1, u_2)$ we actually have to fill in some details. The countable set $X_i \in \mathsf{count}(E(u_i)) \subseteq 2^V$ of children of $[u_i]_{R(M_\equiv)}$ (which is universally quantified in (21)) can be represented as a function $f_i : [|X_i|] \times \mathbb{N} \to \Sigma$ such that the following holds:

$$[\forall j \in [|X_i|] : \mathsf{acc}_{M_E}(u_i \otimes f_i[j])]\ \wedge\ [\forall j, l \in [|X_i|] : j = l \vee \neg \mathsf{acc}_{M_\equiv}(f_i[j] \otimes f_i[l])]\,.$$

Hence, $\forall X_i \in \mathsf{count}(E(u_i)) \cdots$ in (21) can be replaced by:

$$\begin{aligned}
&\forall \kappa_i \in \mathbb{N} \cup \{\aleph_0\}\ \forall f_i : [\kappa_i] \times \mathbb{N} \to \Sigma : \\
&\qquad (\exists j \in [\kappa_i] : \neg \mathsf{acc}_{M_E}(u_i \otimes f_i[j])) \vee \\
&\qquad (\exists j, l \in [\kappa_i] : j \neq l \wedge \mathsf{acc}_{M_\equiv}(f_i[j] \otimes f_i[l]))\ \vee \cdots\,.
\end{aligned}$$

Next, the formula $\exists x, y \in X_1 \cup X_2 : \neg \mathsf{iso}_{k+1}(x, y)$ in (22) can be replaced by:

$$\bigvee_{i \in \{1,2\}} \exists j, l \in [\kappa_i] : \neg \mathsf{iso}_{k+1}(f_i[j], f_i[l])\ \vee\ \exists j \in [\kappa_1]\, \exists l \in [\kappa_2] : \neg \mathsf{iso}_{k+1}(f_1[j], f_2[l]).$$

Similarly, the formula $\exists x \in X_1 \cup X_2\ \exists y \in (E(u_1) \cup E(u_2)) \setminus (X_1 \cup X_2) : \mathsf{iso}_{k+1}(x, y)$ in (23) can be replaced by

$$\begin{aligned}
\bigvee_{i \in \{1,2\}} \exists j \in [\kappa_i]\ \exists v : \mathbb{N} \to \Sigma : \mathsf{iso}_{k+1}(f_i[j], v)\ \wedge& \\
(\mathsf{acc}_{M_E}(u_1 \otimes v) \vee \mathsf{acc}_{M_E}(u_2 \otimes v))\ \wedge& \\
\forall l \in [\kappa_1] : \neg \mathsf{acc}_{M_\equiv}(f_1[l] \otimes v)\ \wedge& \\
\forall l \in [\kappa_2] : \neg \mathsf{acc}_{M_\equiv}(f_2[l] \otimes v)\ .&
\end{aligned}$$

Note that in line (19) and (20) we introduce a new $\forall\exists$ second-order block of quantifiers. The same holds for the rest of the formula: We introduce two universal set quantifiers in (21) followed by the existential quantifier $\exists v : \mathbb{N} \to \Sigma$ in the above formula. Since by induction, iso_{k+1} is a $\Pi^1_{2n-2(k+1)-4}$-statement, it follows that $\mathrm{iso}_k(u_1, u_2)$ is a $\Pi^1_{2n-2k-4}$-statement. □

Corollary 12 and Theorem 28 imply:

Corollary 29. *Assuming* **CH**, *the isomorphism problem for (injectively) ω-automatic trees of finite height is recursively equivalent to the second-order theory of* $(\mathbb{N}, +, \times)$.

Remark 30. *For the case $n = 3$ we can avoid the use of* **CH** *in Theorem 28: Let us consider the proof of Theorem 28 for $n = 3$. Then, the binary relation iso_1 (which holds between two ω-words u, v in F if and only if $[u]$ and $[v]$ are on level 1 and $F(u) \cong F(v)$) is a Π^0_1-predicate. It follows that this relation is Borel (see e.g. [15] for background on Borel sets). Now let u be an ω-word on level 1 in F. It follows that the set of all ω-words v on level 1 with $\mathsf{iso}_1(u, v)$ is again Borel. Now, every uncountable Borel set has cardinality $2^{\aleph_0}$ (this holds even for analytic sets [15]). It follows that the definition of iso_0 in the proof of Theorem 28 is correct even without assuming* **CH**. *Hence, $\mathsf{Iso}(\mathcal{T}_3)$ belongs to Π^1_2 (recall that we proved Π^1_1-hardness for this problem in Section 6), this can be shown in* **ZFC**.

8. Open problems

The main open problem concerns upper bounds in case we assume the negation of the continuum hypothesis. Assuming $\neg$**CH**, is the isomorphism problem for (injectively) ω-automatic trees of height n still analytical? In our paper [21] we also proved that the isomorphism problem for automatic linear orders is Σ^1_1-complete and hence not arithmetical. This leads to the question whether our techniques for ω-automatic trees can be also used for proving lower bounds on the isomorphism problem for ω-automatic linear orders. More specifically, one might ask whether the isomorphism problem for ω-automatic linear orders is analytical. A more general question asks for the complexity of the isomorphism problem for ω-automatic structures in general. On the face of it, it is an existential third-order property (since any isomorphism has to map second-order objects to second-order objects). But it is not clear whether it is complete for this class.

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