Characterizing Probabilistic ω-Recognizability by MSO Logic and Regular Expressions

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1 Introduction

Probabilistic automata, introduced already by Rabin [12], form a flourishing field. Their applications range from speech recognition [13] over prediction of climate parameters [10] to randomized distributed systems [9]. For surveys of theoretical results see the books [11,4]. Recently, the concept of probabilistic automata has been transferred to infinite words by Baier and Grösser [1]. This concept led to further research [2,5,6,7,8,14].

Though probabilistic automata admit a natural quantitative behavior, namely the acceptance probability of each word, the main research interest has been towards qualitative properties (for instance the language of all words with positive acceptance probability). We consider the behavior of a probabilistic automaton as function mapping finite or infinite words to a probability value.

On the other hand, there are two classical characterizations of recognizable languages: Büchi and Elgot established the characterization using monadic second order logic, and Kleene used regular expressions as a model expressively equivalent to recognizable languages. Both results were a paramount success with applications in all fields of theoretical computer science. We establish both results in the context of probabilistic ω-automata.

2 Definitions and Results

For the following, let Σ always denote an alphabet and $w \in \Sigma^\omega$ a word. For a set X, we denote the set of all probability distributions on X by $\Delta(X)$. Given a $p \in (0,1)$, we use the Bernoulli measure B_p on $\{0,1\}^\omega$ which is uniquely defined by $\mathrm{B}_p(u_1 \cdots u_n \{0,1\}^\omega) = p^{|\{i | u_i = 1\}|}(1-p)^{|\{i | u_i = 0\}|}$ for $u_1, \ldots, u_n \in \{0,1\}$. As there is a natural bijection from $\{0,1\}^\omega$ to $2^{\mathbb{N}}$, we regard B_p also as a measure on $2^{\mathbb{N}}$.

Definition 1 (Probabilistic Muller automata). *A* probabilistic Muller automaton *is a tuple $A = (Q, \delta, \mu, \mathrm{Acc})$ where – Q is a non-empty, finite* set of states – $\delta \colon Q \times \Sigma \to \Delta(Q)$ *the* transition probability function – $\mu \in \Delta(Q)$ *the* initial probabilities – $\mathrm{Acc} \subseteq 2^Q$ *a* Muller acceptance condition. *For a word $w = w_1 w_2 \cdots \in \Sigma^\omega$, the behavior of A is defined using the unique measure $\mathbb{P}_A^w$ on Q^ω defined by $\mathbb{P}_A^w(q_0 \cdots q_n Q^\omega) = \mu(q_0) \prod_{i=1}^n \delta(q_{i-1}, w_i, q_i)$. The* behavior $\|A\|$ *of A is then given by $\|A\|(w) = \mathbb{P}_A^w(\rho \in Q^\omega \ ; \ \inf(\rho) \in \mathrm{Acc})$, where* $\inf(\rho)$ *designates the set of states occurring infinitely often in ρ.*

Next, we define the syntax and semantics of probabilistic MSO logic and probabilistic ω-regular expressions.

Definition 2 (Probabilistic MSO logic). *The syntax of a probabilistic MSO φ is given in BNF by*

$$\varphi ::= \psi \mid \varphi \wedge \varphi \mid \neg\varphi \mid \mathbb{E}_p X.\varphi,$$
$$\psi ::= \mathrm{P}_a(x) \mid x \in X \mid x \leq y \mid \psi \wedge \psi \mid \neg\psi \mid \forall x.\psi \mid \forall X.\psi,$$

where ψ denotes a Boolean PMSO formula, *$a \in \Sigma$, and x, y (X) are first-order (resp. second-order) variables.*

The semantics $[\![\varphi]\!]$ are defined inductively: Given a word $w \in \Sigma^\omega$ and an assignment $\alpha \colon \mathcal{V} \to \mathbb{N} \cup 2^{\mathbb{N}}$, for a Boolean formula ψ we define $[\![\psi]\!](w,\alpha) = 1$ if (w,α) satisfies ψ in the classical sense and $[\![\varphi]\!](w,\alpha) = 0$ otherwise. For probabilistic MSO formulas φ the semantics are given by

$$[\![\varphi_1 \wedge \varphi_2]\!](w,\alpha) = [\![\varphi_1]\!](w,\alpha) \cdot [\![\varphi_2]\!](w,\alpha), \qquad [\![\neg\varphi]\!](w,\alpha) = 1 - [\![\varphi]\!](w,\alpha),$$
$$[\![\mathbb{E}_p X.\varphi]\!](w,\alpha) = \int_{2^{\mathbb{N}}} [\![\varphi]\!](w, \alpha[X \mapsto M])\ \mathrm{B}_p(\mathrm{d}M).$$

Definition 3 (Probabilistic ω-regular expressions). *The set $p\omega RE$ of all probabilistic ω-regular expressions is the smallest set satisfying*

- *$\Sigma^\omega \in p\omega RE$*
- *If $\emptyset \neq A \subseteq \Sigma$ and $(E_a)_{a \in A} \in p\omega RE$, then $\sum_{a \in A} aE_a \in p\omega RE$*
- *If $p \in [0,1]$ and $E, F \in p\omega RE$, then $pE + (1-p)F \in p\omega RE$ and $pE \in p\omega RE$*
- *If $E\Sigma^\omega \in p\omega RE$ and $F \in p\omega RE$, then $EF \in p\omega RE$*
- *If $E\Sigma^\omega + F \in p\omega RE$, then $E^*F + E^\omega \in p\omega RE$, $E^\omega \in p\omega RE$ and $E^*F \in p\omega RE$*
- *The set $p\omega RE$ is closed under distributivity of $\cdot$ over $+$, associativity, and commutativity of $+$ and multiplication by real numbers.*

The semantics of probabilistic ω-regular expressions are inductively defined by $\|\Sigma^\omega\|(w) = \mathbb{1}_{\Sigma^\omega}(w)$ and

$$\|a\|(w) = \begin{cases} 1 & \text{if } w = a \\ 0 & \text{otherwise} \end{cases} \qquad \|EF\|(w) = \sum_{uv=w} \|E\|(u) \cdot \|F\|(v)$$
$$\|p\|(w) = \begin{cases} p & \text{if } w = \varepsilon \\ 0 & \text{otherwise} \end{cases} \qquad \|E^*\|(w) = \sum_{u_1 \cdots u_n = w} \|E\|(u_1) \cdots \|E\|(u_n)$$
$$\|E + F\|(w) = \|E\|(w) + \|F\|(w) \qquad \|E^\omega\|(w) = \lim_{n \to \infty} \sum_{u_1 \cdots u_n v = w} \|E\|(u_1) \cdots \|E\|(u_n)$$

where $a \in \Sigma$ and $p \in [0,1]$.

Note that the semantics of $E + F$, E^*, and E^ω are well-defined because of the syntax restrictions imposed on pωRE.

Theorem 4. *Let $f \colon \Sigma^\omega \to [0,1$. The following statements are equivalent*

1. *$f = \|A\|$ for a probabilistic Muller automaton A*
2. *$f = [\![\varphi]\!]$ for a probabilistic MSO sentence φ*
3. *$f = \|E\|$ for a probabilistic ω-regular expression E*

All equivalences given in Theorem 1 are shown using effective constructions. Hence decidability results for probabilistic ω-automata transfer to probabilistic MSO logic and probabilistic ω-regular expressions. For example it is undecidable, given an automaton A, a sentence φ, or an expression E, if there is a word $w \in \Sigma^\omega$ with $\|A\|(w) > 0$, resp. $[\![\varphi]\!](w) > 0$, resp. $\|E\|(w) > 0$.

If an expression E is *ω-deterministic*, i.e. no sub-expression F^ω in E contains probability constants other than 0 or 1, then the above problem is decidable.

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