

Characterizing Tree Valuation Weighted Languages by Multioperator Weighted Tree Languages

Markus Teichmann¹ Johannes Osterholzer²

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Outline

Introduction

Multioperator Weighted Tree Automaton

Tree Valuation Weighted Tree Automaton

Simulation

Construction of the Multioperator-Monoid

Automaton Equivalence

Logics

Multioperator Expressions

Tree Valuation MSO

Transformations

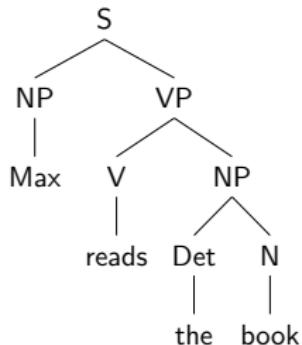
Conclusion & Related Work

Motivation

Tree Languages

- ▶ automated theorem proving,
databases, natural language
processing, XML, etc.
- ▶ recognized/defined by [Büchi/Elgot]
 - ▶ finite state tree automata
 - ▶ monadic second order logic (mso)

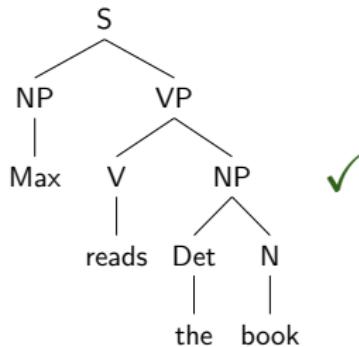
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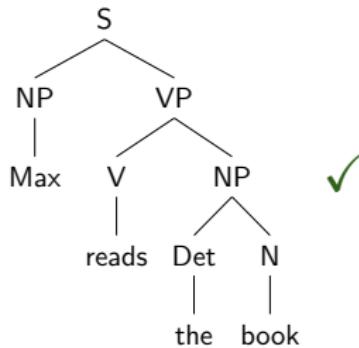
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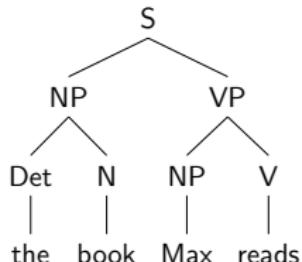
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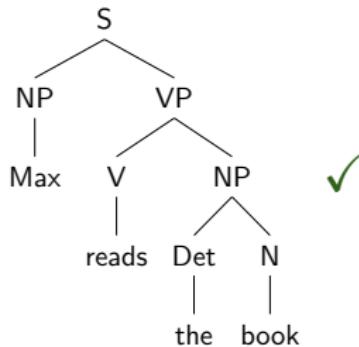
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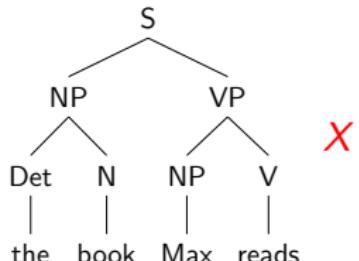


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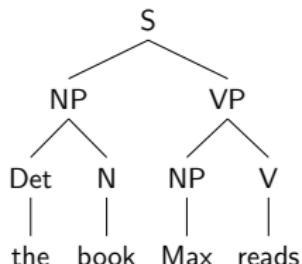
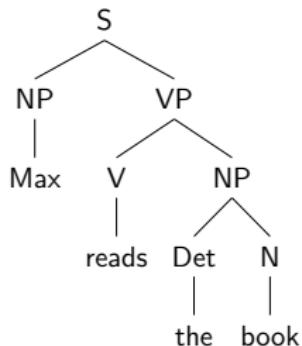


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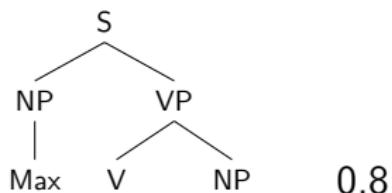
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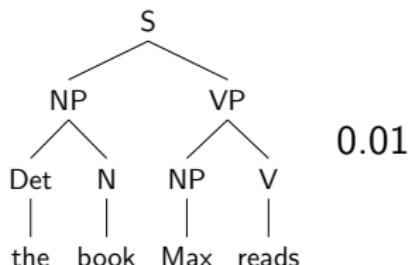
Weighted Tree Languages

- ▶ quantitative aspects
 - ▶ degree of belief
 - ▶ counts of features
- ▶ recognized/defined by
 - ▶ weighted finite state automata
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Motivation



0.8



0.01

Tree Languages

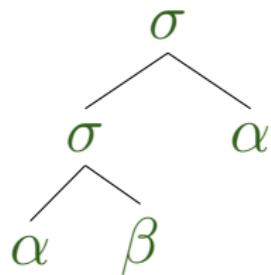
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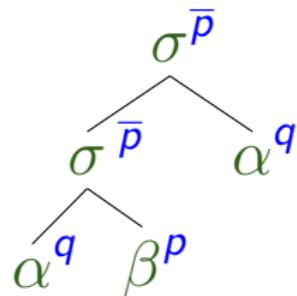
Local State Behavior

- ▶ finite state set $Q = \{\bar{p}, p, q\}$
- ▶ runs $R_Q(\xi) = \{r \mid r: \text{pos}(\xi) \rightarrow Q\}$



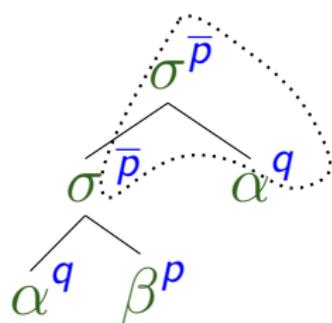
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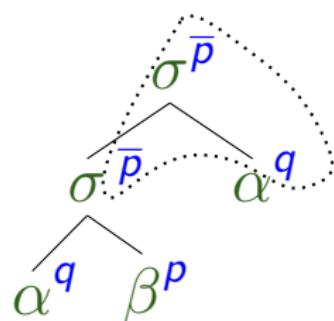


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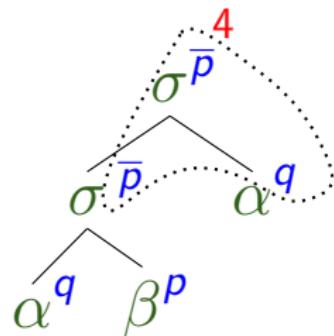


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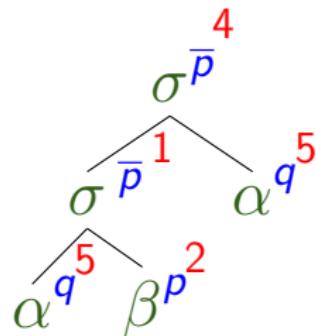
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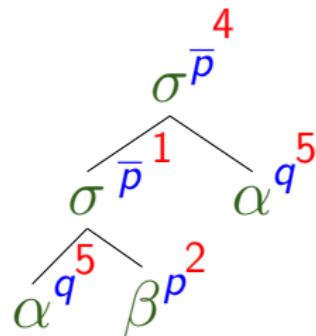
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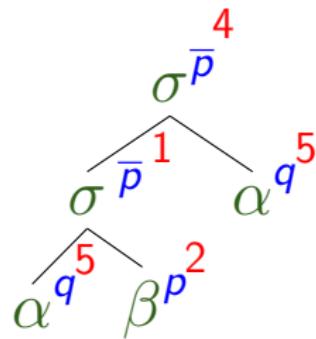
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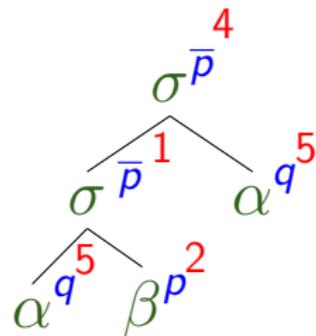
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Semiring: $S_{\mathbb{N}} = (\mathbb{N}, +, \cdot, 0, 1)$

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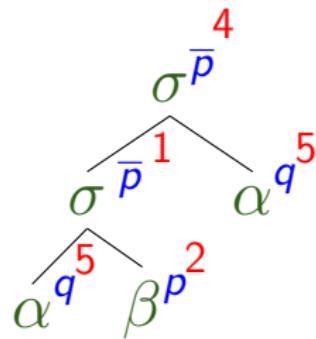


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$$4 \cdot 1 \cdot 5 \cdot 2 \cdot 5 = 200$$

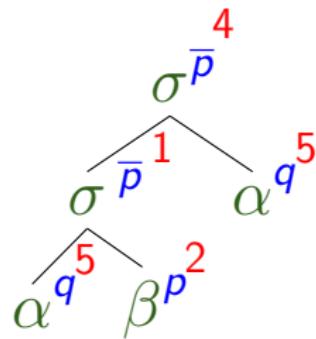
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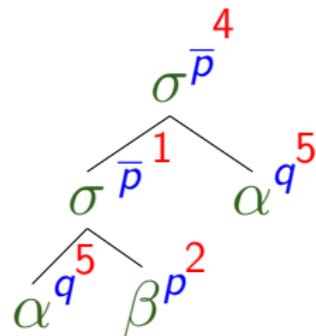


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$$\min\{4, 1, 5, 2, 5\} = 1$$

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- ▶ need for more flexible formalisms

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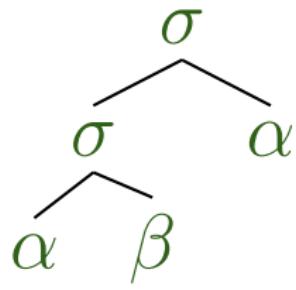
Transformations

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multioperator monoid:

$$\mathcal{A} = (A, +, 0, \Omega)$$

- ▶ $(A, +, 0)$ commutative monoid
- ▶ $\Omega \subseteq \text{Ops}(A)$ - absorptive



m-wta over Σ and \mathcal{A}

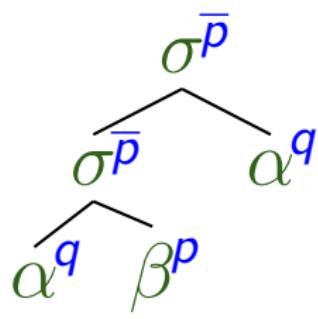
$$\mathcal{M} = (Q, \delta, F)$$

- ▶ $\delta = (\delta_\sigma \mid \sigma \in \Sigma)$
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language recognized by \mathcal{M}

$$[\![\mathcal{M}]\!] : T_\Sigma \rightarrow \mathcal{A}$$

$$[\![\mathcal{M}]\!](\xi) = \sum_{\substack{r \in R_Q(\xi) \\ r(\varepsilon) \in F}} \delta(r, \xi, \varepsilon)$$



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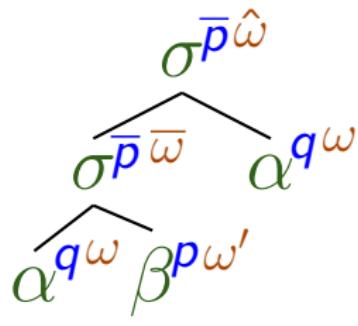
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Multioperator Weighted Tree Automaton

[FSV12]



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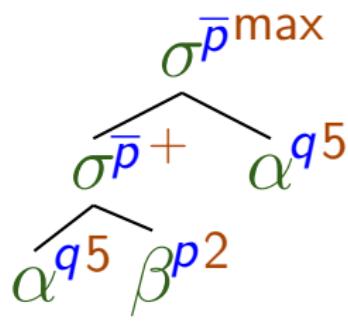
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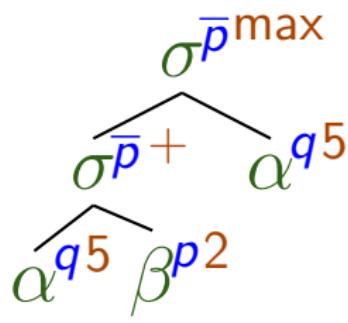
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$$\max(+\left(5, 2\right), 5) = 7$$

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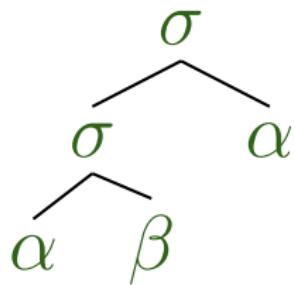
Tree Valuation Weighted Tree Automaton

[DGMM11]

product tree valuation monoid:

$$\mathbb{D} = (\mathcal{D}, +, \diamond, 0, 1, \text{Val})$$

- ▶ $(\mathcal{D}, +, 0)$ commutative monoid
- ▶ $\text{Val}: T_{\mathcal{D}}^u \rightarrow \mathcal{D}$ - only for 0-free
- ▶ \diamond “multiplication” (only for logics)



tv-wta over Σ and \mathbb{D}

$$\mathcal{N} = (Q, \mu, F)$$

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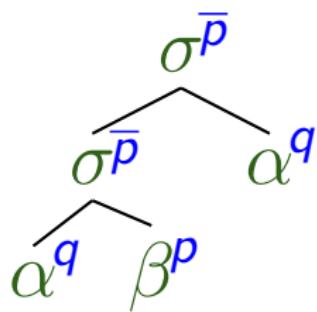
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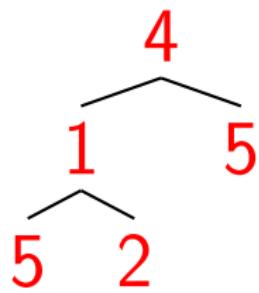
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$$\text{Val} \left(\begin{array}{c} & 4 \\ & / \quad \backslash \\ 1 & & 5 \\ / \quad \backslash \\ 5 & 2 \end{array} \right)$$

tv-wta over Σ and \mathbb{D}

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Theorem

For every regular product tv-monoid $\mathbb{D} = (\mathcal{D}, +, \diamond, \mathbf{0}, \mathbf{1}, \text{Val})$
m-monoid $\mathcal{A} = (\mathcal{A}, \oplus, \mathbf{0}, \Omega)$

tv-wta

m-wta

$\text{Rec}(\Sigma, X)$: class of recognizable weighted tree languages over X
 $\text{Def}(\Sigma, X)$: class of definable weighted tree languages over X

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$$\text{tv-wta} \xrightarrow{\llbracket \cdot \rrbracket} \text{Rec}(\Sigma, \mathbb{D})$$

$$\text{m-wta} \xrightarrow[\llbracket \cdot \rrbracket]{} \text{Rec}(\Sigma, \mathcal{A})$$

$\text{Rec}(\Sigma, X)$: class of recognizable weighted tree languages over X
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$$\text{tv-wta} \xrightarrow{\llbracket \cdot \rrbracket} \text{Rec}(\Sigma, \mathbb{D})$$

$$\text{m-wta} \xrightarrow[\llbracket \cdot \rrbracket]{} \text{Rec}(\Sigma, \mathcal{A}_{\mathbb{D}})$$

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Construction of the m-monoid $\mathcal{A}_{\mathbb{D}}$

$$\mathbb{D} = (\mathcal{D}, +, \diamond, 0, 1, \text{Val}) \rightsquigarrow \mathcal{A}_{\mathbb{D}} = (\mathcal{D} \times T_{\mathcal{D}}^u, \oplus, (0, 0), \Omega)$$

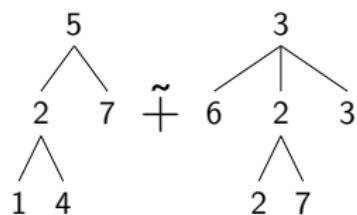
Construction of the m-monoid $\mathcal{A}_{\mathbb{D}}$

$$\mathbb{D} = (D, +, \diamond, 0, 1, \text{Val}) \rightsquigarrow \mathcal{A}_{\mathbb{D}} = (D \times T_D^u, \oplus, (0, 0), \Omega)$$

$$\begin{aligned}\oplus: (D \times T_D^u) \times (D \times T_D^u) &\rightarrow (D \times T_D^u) \\ (\textcolor{red}{d}_1, \xi_1) \oplus (\textcolor{red}{d}_2, \xi_2) &= (\textcolor{red}{d}_1 + \textcolor{red}{d}_2, \xi_1 \tilde{+} \xi_2)\end{aligned}$$

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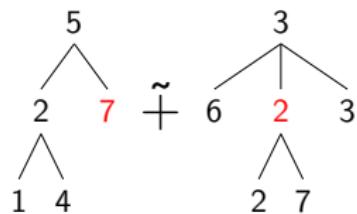
$$\mathbb{D} = (D, +, \diamond, 0, 1, \text{Val}) \rightsquigarrow \mathcal{A}_{\mathbb{D}} = (D \times T_D^u, \oplus, (0, 0), \Omega)$$



$$\oplus: (D \times T_D^u) \times (D \times T_D^u) \rightarrow (D \times T_D^u)$$
$$(d_1, \xi_1) \oplus (d_2, \xi_2) = (d_1 + d_2, \xi_1 \tilde{+} \xi_2)$$

Construction of the m-monoid $\mathcal{A}_{\mathbb{D}}$

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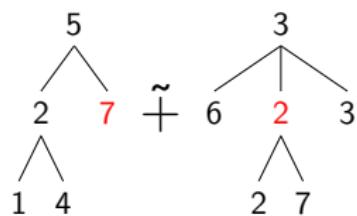


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=

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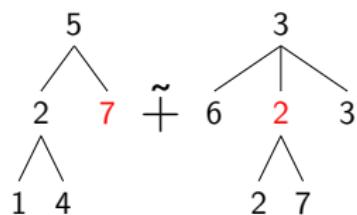


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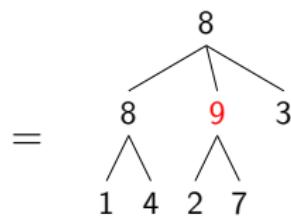
$$= \begin{array}{c} (5 + 3) \\ \diagdown \quad \diagup \\ (2 + 6) \quad (7 + 2) \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad 4 \quad 2 \quad 7 \end{array}$$

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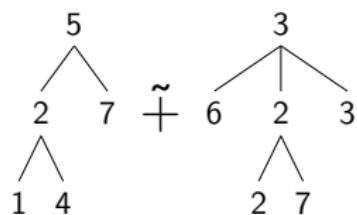


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$$\Omega = \{\text{valtop}_d^{(k)} \mid d \in D, k \in \mathbb{N}\}$$

An equals sign followed by a tree with root 8. This tree has three children: 8, 9, and 3. The child 8 has children 1 and 4. The child 9 has children 2 and 7.

```
graph TD; 8 --- 8; 8 --- 9; 8 --- 3; 8 --- 1; 8 --- 4; 9 --- 2; 9 --- 7;
```

Construction of the m-monoid $\mathcal{A}_{\mathbb{D}}$

$$\mathbb{D} = (\mathcal{D}, +, \diamond, 0, 1, \text{Val}) \rightsquigarrow \mathcal{A}_{\mathbb{D}} = (\mathcal{D} \times T_{\mathcal{D}}^u, \oplus, (0, 0), \Omega)$$

$$\begin{aligned}\oplus: (\mathcal{D} \times T_{\mathcal{D}}^u) \times (\mathcal{D} \times T_{\mathcal{D}}^u) &\rightarrow (\mathcal{D} \times T_{\mathcal{D}}^u) \\ (\mathbf{d}_1, \xi_1) \oplus (\mathbf{d}_2, \xi_2) &= (\mathbf{d}_1 + \mathbf{d}_2, \xi_1 \tilde{+} \xi_2)\end{aligned}$$

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$$\begin{aligned}\oplus: (\mathcal{D} \times T_{\mathcal{D}}^u) \times (\mathcal{D} \times T_{\mathcal{D}}^u) &\rightarrow (\mathcal{D} \times T_{\mathcal{D}}^u) \\ (\textcolor{red}{d}_1, \xi_1) \oplus (\textcolor{red}{d}_2, \xi_2) &= (\textcolor{red}{d}_1 + \textcolor{red}{d}_2, \xi_1 \tilde{+} \xi_2)\end{aligned}$$

$$\Omega = \{\text{valtop}_{\textcolor{brown}{d}}^{(k)} \mid d \in \mathcal{D}, k \in \mathbb{N}\}$$

$$\begin{aligned}\text{valtop}_{\textcolor{brown}{d}}^{(k)}: (\mathcal{D} \times T_{\mathcal{D}}^u)^k &\rightarrow \mathcal{D} \times T_{\mathcal{D}}^u \\ \text{valtop}_{\textcolor{brown}{d}}^{(k)}((\textcolor{red}{d}_1, \xi_1), \dots, (\textcolor{red}{d}_k, \xi_k)) &= (\text{Val}(\xi), \xi) \\ \xi &= \textcolor{brown}{d}(\xi_1, \dots, \xi_k)\end{aligned}$$

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valtop₅⁽²⁾:

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$\text{valtop}_5^{(2)}$:

$$(6, \begin{array}{c} 8 \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array}), (9, \begin{array}{c} 7 \\ | \\ 4 \end{array})$$



$$\begin{array}{c} 5 \\ \diagup \quad \diagdown \\ 8 \quad 7 \\ \diagup \quad \diagdown \\ 2 \quad 3 \quad 4 \end{array}$$

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$$(6, \begin{array}{c} 8 \\ \diagup \quad \diagdown \\ 2 \quad 3 \end{array}), (9, \begin{array}{c} 7 \\ | \\ 4 \end{array})$$



$$\left(\text{Val} \left(\begin{array}{c} 5 \\ \diagup \quad \diagdown \\ 8 \quad 7 \\ \diagup \quad | \quad \diagdown \\ 2 \quad 3 \quad 4 \end{array} \right), \begin{array}{c} 5 \\ \diagup \quad \diagdown \\ 8 \quad 7 \\ \diagup \quad | \quad \diagdown \\ 2 \quad 3 \quad 4 \end{array} \right)$$

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Theorem

For every regular product tv-monoid $\mathbb{D} = (\mathcal{D}, +, \diamond, \mathbf{0}, \mathbf{1}, \text{Val})$

$$\text{tv-wta} \xrightarrow{\llbracket \cdot \rrbracket} \text{Rec}(\Sigma, \mathbb{D})$$

$$\text{m-wta} \xrightarrow[\llbracket \cdot \rrbracket]{} \text{Rec}(\Sigma, \mathcal{A}_{\mathbb{D}})$$

$\text{Rec}(\Sigma, X)$: class of recognizable weighted tree languages over X
 $\text{Def}(\Sigma, X)$: class of definable weighted tree languages over X

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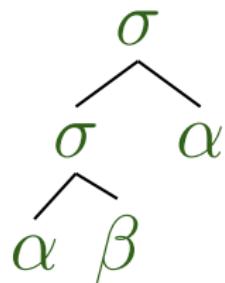
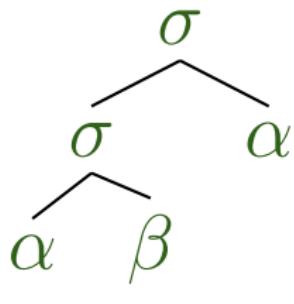
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Automata Equivalence

$$\mathcal{N} = (\textcolor{blue}{Q}, \mu, F) \quad \xleftrightarrow{\text{related}} \quad \mathcal{M} = (\textcolor{blue}{Q}, \delta, \textcolor{blue}{F})$$

tv-wta over \mathbb{D} m-wta over $\mathcal{A}_{\mathbb{D}}$

$$\mu_{\sigma}(q_1 \dots q_k, q) = \textcolor{red}{d} \quad \text{iff} \quad \delta_{\sigma}(q_1 \dots q_k, \textcolor{blue}{q}) = \text{valtop}_{\textcolor{brown}{d}}^{(k)}$$

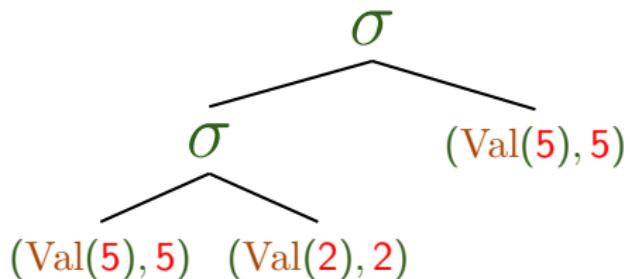
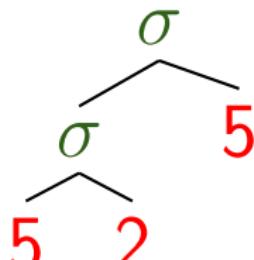


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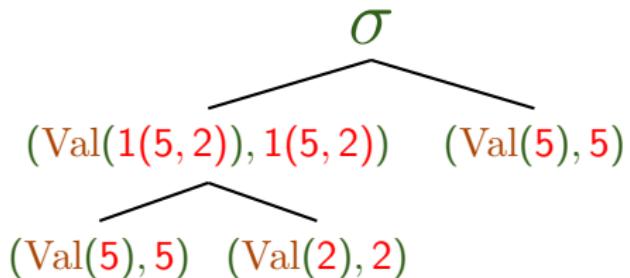
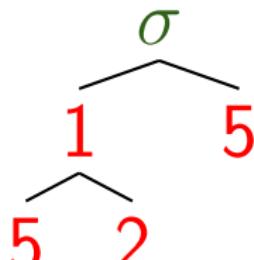


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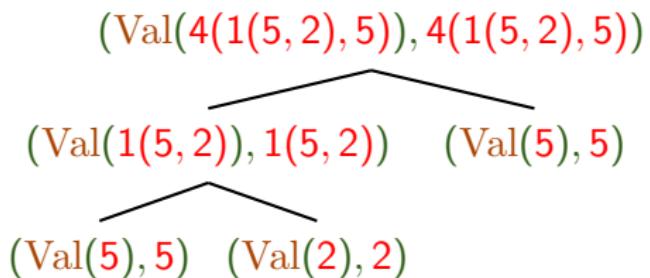
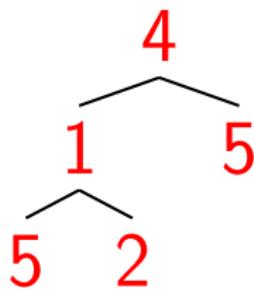
tv-wta over \mathbb{D}

\longleftrightarrow
related

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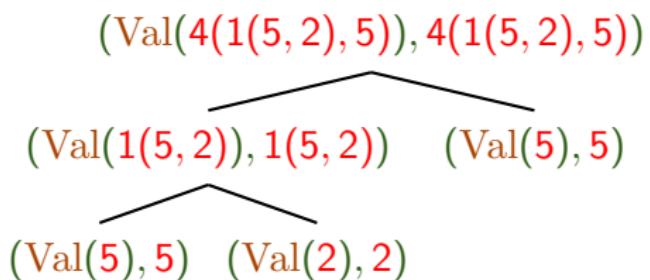
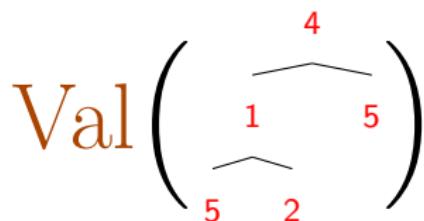


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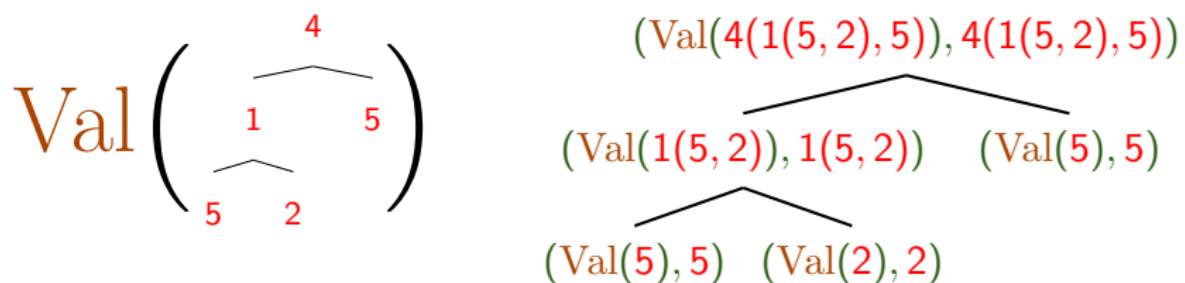


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$$[\![\mathcal{N}]\!] = \pi_1^2([\![\mathcal{M}]\!])$$

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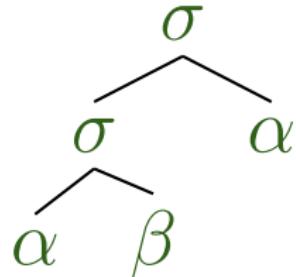
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tv-mso formulas m-expressions

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Multioperator Expressions

[FSV12]



m-expressions over Σ and m-monoid \mathcal{A} :

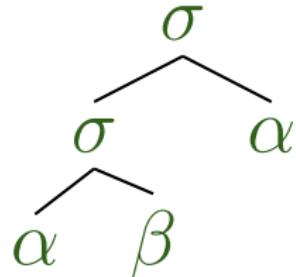
- ▶ $H(\omega)$ where $\omega = (\omega_\sigma \in \Omega \mid \sigma \in \Sigma_{\mathcal{U}})$
- ▶ $(e_1 + e_2)$
- ▶ $\sum_x e$, $\sum_{\mathcal{X}} e$
- ▶ $(\varphi \triangleright e)$ where $\varphi \in \text{MSO}(\Sigma)$

language defined by e : $\llbracket e \rrbracket : T_\Sigma \rightarrow \mathcal{A}$

class of m-expression definable tree languages: $\text{Def}(\Sigma, \mathcal{A})$

Multioperator Expressions

[FSV12]



$$\sum_{\textcolor{blue}{x}} (\text{root}(\textcolor{blue}{x}) \triangleright H(\omega))$$

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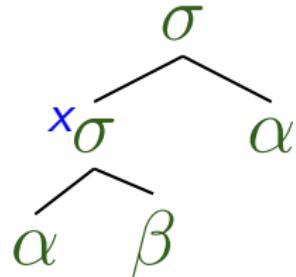
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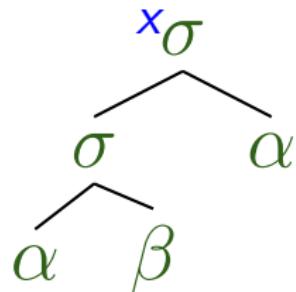
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[FSV12]



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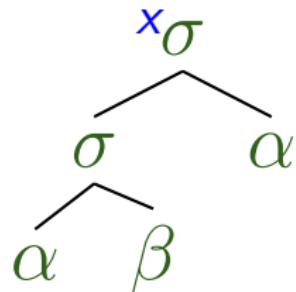
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Multioperator Expressions

[FSV12]



$$\sum_x (\text{root}(x) \triangleright H(\omega))$$

$$\omega_{(\alpha, \emptyset)}() = 5$$

$$\omega_{(\beta, \emptyset)}() = 2$$

$$\omega_{(\sigma, \emptyset)}(n_1, n_2) = n_1 + n_2$$

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m-expressions over Σ and m-monoid \mathcal{A} :

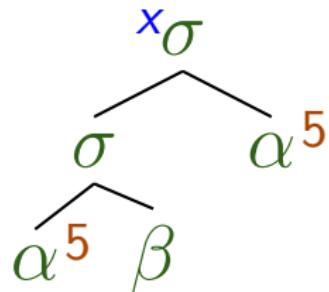
- ▶ $H(\omega)$ where $\omega = (\omega_\sigma \in \Omega \mid \sigma \in \Sigma_U)$
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Multioperator Expressions

[FSV12]



$$\sum_x (\text{root}(x) \triangleright H(\omega))$$

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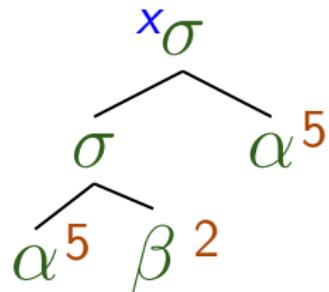
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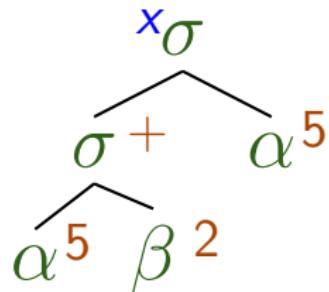
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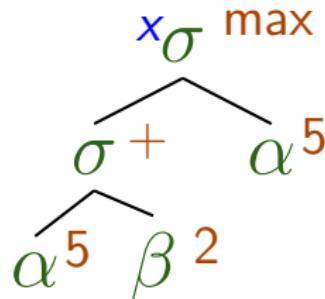
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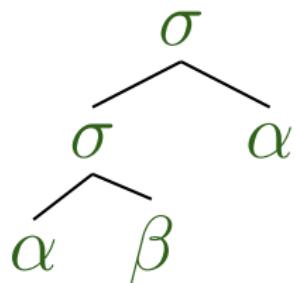
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Tree Valuation MSO

[DGMM11]



$$\mathbb{D} = (\mathcal{D}, +, \diamond, 0, 1, \text{Val})$$

tv-mso over Σ and ptv-monoid \mathbb{D} :

$$\varphi ::= d \mid \beta \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \exists x.\varphi \mid \forall x.\varphi \mid \exists X.\varphi$$

$\beta \in \text{MSO}(\Sigma)$ - Boolean formula

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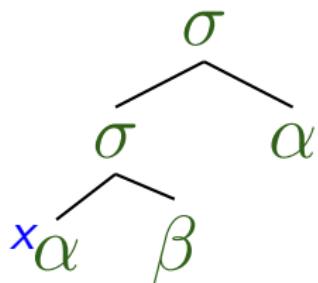
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Example:

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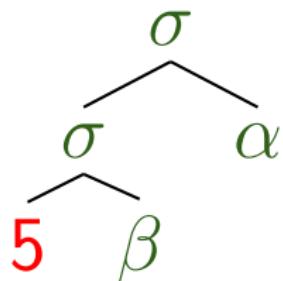
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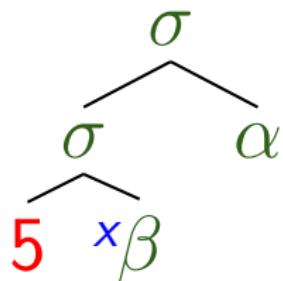
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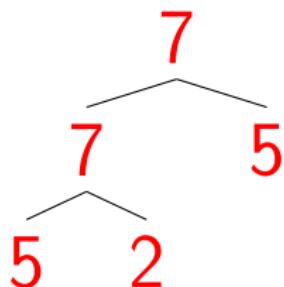
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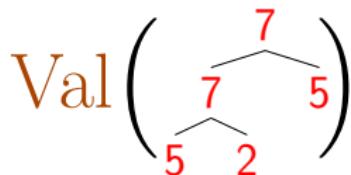
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Theorem

For every regular product tv-monoid $\mathbb{D} = (\mathcal{D}, +, \diamond, \mathbf{0}, \mathbf{1}, \text{Val})$

$$\begin{array}{ccc} \text{tv-wta} & \xrightarrow{[\cdot]} & \text{Rec}(\Sigma, \mathbb{D}) \\ \uparrow \text{related} & & \parallel \\ \text{m-wta} & \xrightarrow{[\cdot]} & \pi_1^2(\text{Rec}(\Sigma, \mathcal{A}_{\mathbb{D}})) \end{array}$$

tv-mso formulas m-expressions

$\text{Rec}(\Sigma, X)$: class of recognizable weighted tree languages over X
 $\text{Def}(\Sigma, X)$: class of definable weighted tree languages over X

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$$\begin{array}{ccccc} \text{tv-wta} & \xrightarrow{\llbracket \cdot \rrbracket} & \text{Rec}(\Sigma, \mathbb{D}) & \xlongequal{\text{[DGMM11]}} & \text{Def}(\Sigma, \mathbb{D}) & \xleftarrow{\llbracket \cdot \rrbracket} & \text{tv-mso formulas} \\ \updownarrow \text{related} & & \parallel & & & & \\ \text{m-wta} & \xrightarrow{\llbracket \cdot \rrbracket} & \pi_1^2(\text{Rec}(\Sigma, \mathcal{A}_{\mathbb{D}})) & = & \pi_1^2(\text{Def}(\Sigma, \mathcal{A}_{\mathbb{D}})) & \xleftarrow{\llbracket \cdot \rrbracket} & \text{m-expressions} \\ & & \text{[FSV12]} & & & & \end{array}$$

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Tree Valuation MSO \rightsquigarrow Multioperator Expressions

$t(\exists x.([\text{root}(x) \wedge 2] \vee 5))$

Tree Valuation MSO \rightsquigarrow Multioperator Expressions

$$\textcolor{red}{t}(\exists x.([\text{root}(x) \wedge 2] \vee 5))$$

$$= \sum_x (\textcolor{red}{t}([\text{root}(x) \wedge 2] \vee \textcolor{blue}{5}))$$

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$$\begin{aligned} & \textcolor{red}{t}(\exists x.([\text{root}(x) \wedge 2] \vee 5)) \\ &= \sum_x (\textcolor{red}{t}([\text{root}(x) \wedge 2] \vee 5)) \\ &= \sum_x (\textcolor{red}{t}(\text{root}(x) \wedge 2) + \textcolor{red}{t}(5)) \end{aligned}$$

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For every $d \in D$:

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$$\begin{array}{ccccc} & & [\text{DGMM11}] & & \\ \text{tv-wta} & \xrightarrow{\llbracket \cdot \rrbracket} & \text{Rec}(\Sigma, \mathbb{D}) & \xlongequal{\quad} & \text{Def}(\Sigma, \mathbb{D}) \xleftarrow{\llbracket \cdot \rrbracket} \text{tv-mso formulas} \\ \updownarrow \text{related} & & \parallel & & \updownarrow t(\cdot) \\ \text{m-wta} & \xrightarrow{\llbracket \cdot \rrbracket} & \pi_1^2(\text{Rec}(\Sigma, \mathcal{A}_{\mathbb{D}})) & = & \pi_1^2(\text{Def}(\Sigma, \mathcal{A}_{\mathbb{D}})) \xleftarrow{\llbracket \cdot \rrbracket} \text{m-expressions} \\ & & [\text{FSV12}] & & \end{array}$$

$\text{Rec}(\Sigma, X)$: class of recognizable weighted tree languages over X
 $\text{Def}(\Sigma, X)$: class of definable weighted tree languages over X

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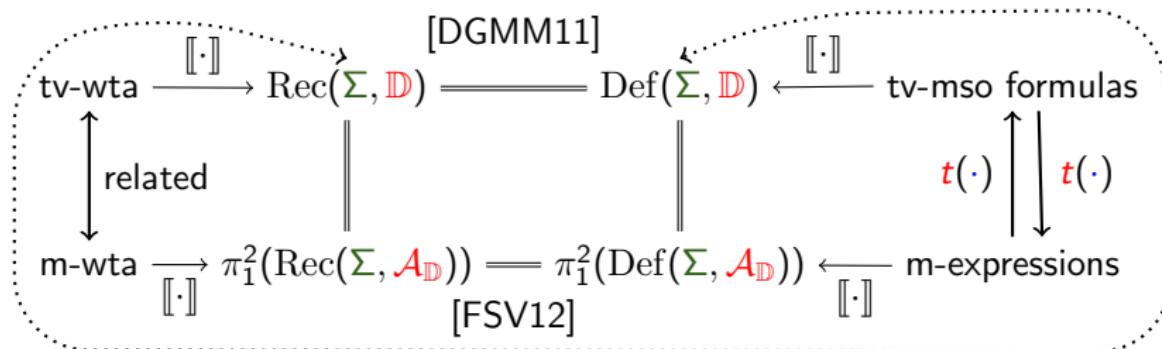
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Related Work

- ▶ Transformations for different logic restrictions
 - ▶ simplifies constructions for some cases
- ▶ Simulation of M-Languages by TV-Languages
 - ▶ based on [Droste, Götze, Märcker, Meinecke (2011)]
 - ▶ no equality, i.e. $\text{Rec}(\Sigma, \mathcal{A}) \subset \text{Rec}(\Sigma, \mathbb{D}_{\mathcal{A}})$

List of References

-  Manfred Droste, Doreen Götze, Steffen Märcker, and Ingmar Meinecke. "Weighted Tree Automata over Valuation Monoids and Their Characterization by Weighted Logics". In: *Algebraic Foundations in Computer Science*. Ed. by Werner Kuich and George Rahonis. Vol. 7020. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 2011, pp. 30–55.
-  Zoltán Fülöp, Torsten Stüber, and Heiko Vogler. "A Büchi-like theorem for weighted tree automata over multioperator monoids". In: *Theory of Computing Systems* 50(2) (2012). Published online 28.10.2010, pp. 241–278.

Multioperator Expressions \rightsquigarrow Tree Valuation MSO

- $t(H(\omega)) = \forall x.\psi^{H(\omega)}$, where $\omega_{(\sigma, U)} = \text{valtop}_d^{(k)}$

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- $t(\varphi \triangleright e') = \varphi \wedge t(e')$.

Tree Valuation MSO \rightsquigarrow Multioperator Expressions

- ▶ Boolean formula β : $t(\beta) = \beta \triangleright t(1)$.
- ▶ $d \in D$: simulates automaton that recognizes d for any input.
- ▶ $t(\varphi_1 \vee \varphi_2) = t(\varphi_1) + t(\varphi_2)$.
- ▶ $\varphi = \varphi_1 \wedge \varphi_2$ is strongly \wedge -restricted:
 - ▶ φ_1 (or φ_2) Boolean: $t(\varphi_1 \wedge \varphi_2) = \varphi_1 \triangleright t(\varphi_2)$
 - ▶ φ_1 and φ_2 almost Boolean, i.e. $\text{step}(\varphi_i) = (a_1^i, \psi_1^i) \dots (a_n^i, \psi_n^i)$
then $t(\varphi_1 \wedge \varphi_2) = \sum_{i \in [n]}^{+} (\psi_i^1 \wedge \psi_j^2) \triangleright t(a_i^1 \diamond a_j^2)$
- ▶ $t(\exists x. \psi) = \sum_x t(\psi)$
- ▶ $t(\exists X. \psi) = \sum_X t(\psi)$
- ▶ $\varphi = \forall x. \psi$ is \forall -restricted: ψ almost Boolean, i.e.
 $\text{step}(\psi) = (d_1, \psi_1) \dots (d_n, \psi_n)$. Let $U = \{X_1, \dots, X_n\}$ and
 $(\omega^\psi)_{(\sigma, U)} = \text{valtop}_{d_U}^{(k)}$ where $d_U = \sum_{\substack{i \in [n] \\ X_i \in U}} d_i$

$$t(\forall x. \psi) = \sum_{X_1} \dots \sum_{X_n} (\forall x. (\bigwedge_{i \in [n]} (x \in X_i) \Leftrightarrow \psi_i)) \triangleright H(\omega^\psi)$$

The Automaton \mathcal{N}_d

- ▶ Runs of the automaton are simulated by a partition of variables $X_{q_1 \dots q_k, \sigma, q}$ representing the transition taken at each position.
- ▶ $\varphi_{\text{partition}}$ checks for a valid run representation
- ▶ $\varphi_{\text{final}} = \forall z. (\text{root}(z) \Rightarrow \bigwedge_{\substack{q_1 \dots q_k, \sigma, q \\ q \notin F}} (z \notin X_{q_1 \dots q_k, \sigma, q}))$
- ▶ We construct: $t(d) = \sum_{X_1} \dots \sum_{X_n} (\varphi_{\text{partition}} \wedge \varphi_{\text{final}}) \triangleright H(\omega^d)$
where $(\omega^d)_{(\sigma, \{X_{q_1 \dots q_k, \sigma, q}\})} = \text{valtop}_{\mu_\sigma(q_1 \dots q_k, q)}$