Recognisability and Algebras of Infinite Trees

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Algebraic Language Theory

Recognisability

φ : free algebra → finite algebra

$L = \varphi^{-1}[P]$

Which algebras?

finite words	monoid/semigroup
infinite words	ω-semigroup
finite trees	clones, preclones, term algebras, forest algebras,…
infinite trees	?

ω-semigroups

$\langle S, S_\omega \rangle$ with associative operations

- $S \times S \to S$
- $S \times S_\omega \to S_\omega$
- $S^\omega \to S_\omega$

Example

$S := [2]$, $S_\omega := [2]$ with products

- $S \times S \to S \quad : (s,t) \mapsto \max\{s,t\}$
- $S \times S_\omega \to S_\omega : (s,u) \mapsto u$
- $S^\omega \to S_\omega \quad : (s_n)_n \mapsto \limsup_{n\to\infty} s_n$

recognises the set of all ω-words containing infinitely many letters a.

($a \mapsto 1$ and $b \mapsto 0$)

Wilke algebras

Replace infinite product $S^\omega \to S_\omega$ by ω-power operation $S \to S^\omega$.

Theorem The infinite product is uniquely determined by the ω-power operation.

Proof Every infinite product $\pi(a_0, a_1, \dots)$ has a factorisation

$$\underbrace{(a_0 \cdots a_{k-1})}_{s} \cdot \underbrace{(a_k \cdots a_{l-1})}_{u} \cdot \underbrace{(a_l \cdots a_{m-1})}_{u} \cdots$$

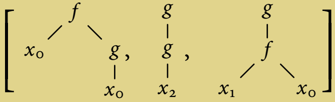
(Theorem of Ramsey). Hence,

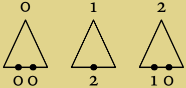
$$\pi(a_0, a_1, \dots) = s \cdot u^\omega.$$

Algebras for infinite trees

Term algebra

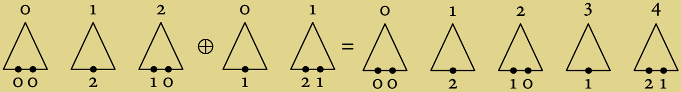
Elements: tuples of finite and infinite terms with variables $x_0, x_1, \ldots$



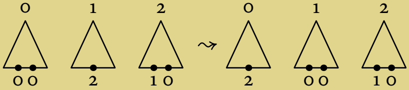


Operations

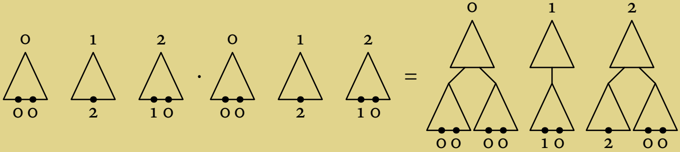
- **horizontal product:** concatenation



- **reordering**



- **vertical product:** substitution



- **infinite product**

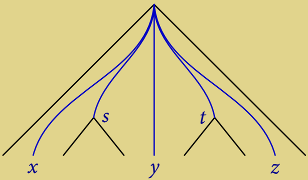
Hyperclones: homomorphic images of the term algebra

Automata to hyperclones

Path-hyperclones

hyperclones associated with an ω-semigroup $\langle S, S_\omega \rangle$

Elements: $\wp(S^n \times \wp(S_\omega))^{<\omega}$



$(s, t, \{x, y, z\})$

Example

$S := [2],\ S_\omega := [2]$ with max as product.

The corresponding hyperclone recognises the set of all trees containing at least one vertex with label a. ($a \mapsto 1$ and $b \mapsto 0$)

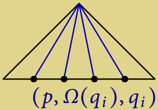
ω-semigroup for an automaton

$$S := Q \times D \times Q \cup \bot \qquad S_\omega := Q \cup \bot$$

$$(p, d, q) \cdot (p', d', q') := \begin{cases} (p,\ \min\{d, d'\},\ q') & \text{if } q = p' \\ \bot & \text{otherwise} \end{cases}$$

homomorphism

transition $(p, a, q_0, \dots, q_{n-1})$

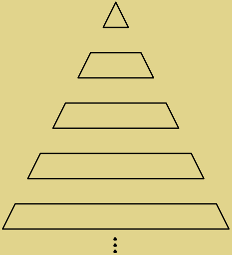


Theorem

The hyperclone associated with an automaton $\mathcal{A}$ recognises $L(\mathcal{A})$.

Hyperclones to automata

Problem: To evaluate an infinite product



an automaton needs to access each “slice”.

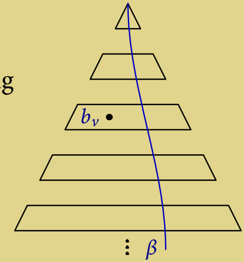
Idea: Compute infinite products one branch at a time.

Fix a branch β.

For each $v \notin \beta$, guess the value b_v of the subtree.

Insert these values and compute the remaining product.

$\mathrm{Tr}(\beta)$: the values collected in this way



Definition

A hyperclone is **path-continuous** if $\pi(a_0, a_1, \dots)$ is determined by $\{\, \mathrm{Tr}(\beta) \mid \beta \text{ a branch} \,\}$.

Examples

(a) Every path-hyperclone is path-continuous.

(b) The following hyperclone is not path-continuous:

Elements: $[4]^n$

$$a \cdot \bar{b} := \max\{a, b_0, \dots, b_{n-1}\}$$

$$\pi(a_0, a_1, \dots) := \max(\{x\} \cup \{ a_i \mid i < \omega \})$$

$$x := \begin{cases} 3 & \text{if there are infinitely many branches from which} \\ & \text{you can always reach a value } {>}0 \\ 2 & \text{if there are finitely many such branches} \\ 0 & \text{if there are no such branches} \end{cases}$$

Theorem

Let L be a set of infinite trees. The following statements are equivalent:

(1) L is recognised by a **tree automaton.**

(2) L is recognised by a **finitary, path-continuous hyperclone.**

(3) L is recognised by a **path-hyperclone** associated with a finite ω-semigroup.

Wilke algebras

Problem

To specify a hyperclone we need an infinite amount of data:

- infinitely many sorts
- infinite product

Solution

- We only need to consider finitely many sorts at a time.
- We can replace π by the ω-power $^\omega$: $a^\omega := \pi(a, a, \dots)$

ω-power

Obviously π determines ${}^{\omega}$.

Question: what about the converse?

Theorem

For finitary, path-continuous hyperclones, ${}^{\omega}$ determines π.

Proof idea

We have to evaluate $\pi(a_0, a_1, \dots)$ using only $\cdot$ and ${}^{\omega}$.

Find a regular sequence $a'_0, a'_1, \dots$ such that

$$\pi(a_0, a_1, \dots) = \pi(a'_0, a'_1, \dots).$$

We can represent $a'_0, a'_1, \dots$ as $u \cdot v^{\omega}$ and set

$$\pi(a_0, a_1, \dots) := u \cdot v^{\omega}.$$

Path labellings

Additive labelling

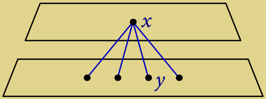
T tree, $\langle S, S_\omega \rangle$ ω-semigroup, $\lambda : \{ (x,y) \in T^2 \mid x < y \} \to S$

$$\lambda(x,z) = \lambda(x,y) \cdot \lambda(y,z) \quad \text{for } x < y < z$$

Labelling for a hyperclone

If y is the k-th successor of x define

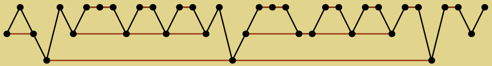
$$\lambda(x,y) := \{ a \cdot_k b \mid a \text{ the element at } x, b \text{ arbitrary} \}$$



Ramseyan splits

λ additive labelling of T, $\sigma : T \to [k]$

- $x \sqsubseteq_\sigma y$ iff $x \leq y$ and $\sigma(x) = \sigma(y) \leq \sigma(z)$, for all $x \leq z \leq y$



- σ is a **Ramseyan split** if

  $\lambda(u, v) \cdot \lambda(x, y) = \lambda(u, v)$ for all $u \sqsubseteq_\sigma x \sqsubset_\sigma y$, $u \sqsubset_\sigma v$ with v, y comparable

Theorem (Colcombet)

Every additive labelling λ has a Ramseyan split $\sigma : T \to [k]$ with $k \leq |S|$.

Theorem

Let λ be an additive labelling of T. There exists a prefix $T_0 \subseteq T$ of bounded height (in terms of S) with back-edges such that

$$\{\, \lambda(\beta) \mid \beta \text{ a branch of } T_0 \,\} = \{\, \lambda(\beta) \mid \beta \text{ a branch of } T \,\}.$$

The syntactic congruence

Definition

$$a \sim_L b \quad :\text{iff} \quad \begin{cases} x \oplus yaz \in L & \Leftrightarrow \quad x \oplus ybz \in L \\ x \oplus y \cdot (a \oplus z)^\omega \in L & \Leftrightarrow \quad x \oplus y \cdot (b \oplus z)^\omega \in L \end{cases}$$

Theorem

$\sim_L$ is the coarsest congruence saturating L.

Theorem

If $\mathfrak{C}$ is finitary and path-continuous, $\sim_P$ is decidable.

Hyperclones and monadic second-order logic

Lemma

The class of languages recognised by finitary, path-continuous hyperclones is effectively closed under

- boolean operations,
- projection.

Theorem

For every MSO-formula φ, we can effectively construct a hyperclone recognising $L(\varphi)$.

Corollary (Rabin)

It is decidable whether an MSO-formula has a tree model.

Summary

- equivalence automata $\leftrightarrow$ hyperclones
- Wilke algebras
- syntactic congruence
- translation MSO $\mapsto$ hyperclones

To do

- simplify definitions
- effective characterisations
- pseudo-varieties