Innerer palindromischer Abschluss

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27. September 2013

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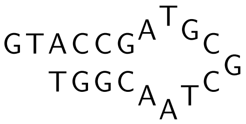

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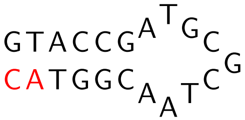

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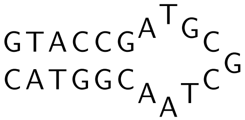
GTACCGATGCGCTAACGGTAC→ 

$xYz\overline{Y}^R \rightarrow xYz\overline{Y}^R\overline{x}^R$

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hairpin completion

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CATGGC$_{A}$ $_{A}$ T C ^{G}

$$xY z \overline{Y}^R \to xY z \overline{Y}^R \overline{x}^R$$
$$Y z \overline{Y}^R x \to \overline{x}^R Y z \overline{Y}^R x$$

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X

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$X \rightarrow XX$

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GTACCG A T G C
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$xYz\overline{Y}^R \rightarrow xYz\overline{Y}^R\overline{x}^R$ hairpin completion

$Yz\overline{Y}^R x \rightarrow \overline{x}^R Yz\overline{Y}^R x$

$yXz \rightarrow yXXz$ duplication languages

Basics

palindrome: $w = w_1 \ldots w_n = w_n \ldots w_1 = w^R$ (*rentner*)

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For a word u, the left (right) palindromic closure of u is a palindrome vu (uv) with v non-empty, such that any other palindrome with u as proper suffix (prefix) has length greater than $|uv|$.

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$aaa \rightarrow \{aaaa\}$

Definitions

Definition

For a word w, the left (right) inner palindromic closure of w, denoted by $\spadesuit_\ell(w)$ ($\spadesuit_r(w)$), is the set of all words $xvuy$ ($xuvy$) for any factorisation $w = xuy$ with possibly empty x, y and non-empty u, v, such that vu (uv) is the left (right) palindromic closure of u. The inner palindromic closure $\spadesuit(w)$ is the union of $\spadesuit_\ell(w)$ and $\spadesuit_r(w)$.

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Definition

For a language L, let $\spadesuit(L) = \bigcup_{w \in L} \spadesuit(w)$. We set $\spadesuit^0(L) = L$, $\spadesuit^n(L) = \spadesuit(\spadesuit^{n-1}(L))$ for $n \geq 1$, $\spadesuit^*(L) = \bigcup_{n \geq 0} \spadesuit^n(L)$.

Observations

Lemma

For every word w, if $u \in \spadesuit^*(w)$, then $w \preccurlyeq u$.

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Lemma

[Propagation rule] For a word $w = a^n b^m$ with positive integers n and m, the set $\spadesuit(w)$ contains all words of length $n + m + 1$ with a letter $x \in \{a, b\}$ inserted at any position i of w, where $0 \leq i < n + m$.

Corollary

For any binary words w and u, $w \preccurlyeq u$ if and only if $u \in \spadesuit^*(w)$.

Regularity of closure

Theorem

The iterated inner palindromic closure of a binary language is regular.

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Theorem (Higman)

If L is a language such that any two words in L are incomparable with respect to the scattered factors partial order, then L is finite.

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Theorem

The finite inner palindromic closure of a regular language is not necessarily regular.

Duplication idea

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Take prefixes of an infinite square-free word, and show that they are from different equivalent classes.

Why it does not work?!

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Why it does not work?!

$\color{red}{abc}\underline{d}abfa$

Why it does not work?!

$abc\underline{d}abfa$

$abcba\underline{d}abfa$

Why it does not work?!

$abc\underline{d}abfa$

$abc\color{red}{ba\underline{d}abfa}$

Why it does not work?!

abcdabfa

abcbadabfa

abcafbadabfa

Bigger alphabets

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Consider now the word abc and the language $(abc)^*$ that contains no palindromes of length greater than one. However, $babc \in \spadesuit(abc)$ can generate at the beginning as many abc's as we want, $(abc)^*babc$.

Bigger alphabets

Consider now the word *abc* and the language $(abc)^*$ that contains no palindromes of length greater than one. However, $babc \in ♠(abc)$ can generate at the beginning as many *abc*'s as we want, $(abc)^* babc$.

Lemma

Let $\Sigma = \{a_1, a_2, \ldots, a_k\}$ *and define the recursive sequences*

$$w'_0 = \varepsilon \text{ and } w_0 = \varepsilon,$$
$$w'_i = w_{i-1} w'_{i-1} \text{ and } w_i = w'_i a_i \text{ for } 1 \leq i \leq k.$$

Then for $1 \leq i \leq k$, $alph(w_i)^* w_i \subseteq ♠^*(w_i)$.

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Then for $1 \leq i \leq k$, $alph(w_i)^* w_i \subseteq \spadesuit^*(w_i)$.

$$a, ab, abac, abacabad, abacabadabacabae, \ldots$$

Parametrized inner closure

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Definition

For a word u and $m, n \in \mathbb{N}$, we define the sets

$$L_{m,n}(w) = \{u \mid u = u^R, u = xw \text{ for } x \neq \varepsilon, |x| \geq n,\ m \geq |w| - |x| \geq 0\},$$
$$R_{m,n}(w) = \{u \mid u = u^R, u = wx \text{ for } x \neq \varepsilon, |x| \geq n,\ m \geq |w| - |x| \geq 0\}.$$

The left (right) (m, n)-palindromic closure of w is the shortest word of $L_{m,n}(w)$ (resp., $R_{m,n}(w)$), or undefined if $L_{m,n}(w)$ (resp., $R_{m,n}(w)$) is empty.

Definition

For non-negative integers n, m with $n > 0$, we define the $\spadesuit_{(m,n)}$ one step inner palindromic closure of some word w as

$$\spadesuit_{(m,n)}(w) = \{u \mid u = xy'z,\ w = xyz,\ \text{and } y' \text{ is obtained by left or right } (m,n)\text{-palindromic closure from } y\}.$$

Example

aab

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$L_{(1,2)}$ $R_{(1,2)}$

$\spadesuit_{(1,2)}(aab) = \{aaaab,$

Example

$R_{(1,2)}$

$\spadesuit_{(1,2)}(aab) = \{aaaab, aabba,$

Example

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$$\spadesuit_{(1,2)}(aab) = \{aaaab, aabba, abaab, baaab, aabaa\}$$

$$\spadesuit(aab) = \{aaab,\ aabb,\ baab,\ aabaa,\ aaba,\ abab\}$$

Example

$$\spadesuit_{(1,2)}(aab) = \{aaaab, aabba, abaab, baaab, aabaa\}$$

$$\spadesuit(aab) = \{aaab, aabb, baab, aabaa, aaba, abab\}$$

Note that $L_{m,n}(w)$ and $R_{m,n}(w)$ are empty if and only if $|w| < n$.

Initial results

Proposition

For any word w with $|w| \geq n$ and positive integer m, the language $\spadesuit^*_{(m,n)}(w)$ is dense with respect to the alphabet $alph(w)$.

Lemma

Let Σ be an alphabet with $||\Sigma|| \geq 2$, $a \notin \Sigma$, and m and n positive integers. Let $w = a^m y_1 a \cdots y_{p-1} a y_p$ be a word such that $alph(w) = \Sigma \cup \{a\}$, $m, p > 0$, $y_i \in \Sigma^*$ for $1 \leq i \leq p$, $|y_1| > 0$, and such that there exists $1 \leq j \leq p$ with $|y_j| \geq n$. Then, for each $v \in \Sigma^*$ with $|v| \geq n$, there exists $w' \in \spadesuit^*_{(m,n)}(w)$ such that v is a prefix of w' and $|w'|_a = |w|_a$.

Main result

Theorem

Let $m > 0$ and $k \geq 2$ be two integers and define $n = \max\{\frac{q_k}{2}, p_k\}$. Let Σ be a k-letter alphabet with $a \notin \Sigma$ and $w = a^m y_1 a y_2 \cdots a y_{r-1} a y_r$ be a word such that $alph(w) = \Sigma \cup \{a\}$, $r > 0$, $y_i \in \Sigma^*$ for all $1 \leq i \leq r$, and there exists j with $1 \leq j \leq r$ and $|y_j| \geq n$. Then $\spadesuit^*_{(m,n)}(w)$ is not regular.

Consequence

The following theorem follows immediately from the previous results.

Theorem

Let $w = a^p y_1 a \cdots y_{r-1} a y_r$, where $a \notin alph(y_i)$ for $1 \leq i \leq r$.
(1) If $\|alph(w)\| \geq 3$ and $|y_j| \geq 3$ for some $1 \leq j \leq r$, then for every positive integer $m \leq p$ we have that $\spadesuit^*_{(m,3)}(w)$ is not regular.
(2) If $\|alph(w)\| \geq 4$ and $|y_j| \geq 2$ for some $1 \leq j \leq r$, then for every positive integer $m \leq p$ we have that $\spadesuit^*_{(m,2)}(w)$ is not regular.
(3) If $\|alph(w)\| \geq 5$, then for every positive integer $m \leq p$ we have that $\spadesuit^*_{(m,1)}(w)$ is not regular.
(4) For every positive integers m and n there exists u with $\spadesuit^*_{(m,n)}(u)$ not regular.

Proof

Theorem (Proof of (1))

There exist infinitely long binary words avoiding both palindromes of length 6 and longer, and squares of words with length 3 and longer.

Proof

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There exist infinitely long binary words avoiding both palindromes of length 6 and longer, and squares of words with length 3 and longer.

Rampersad, Shallit, and Wang gave an infinite word w, that is square-free and has no factors from the set
$\{ac, ad, ae, bd, be, ca, ce, da, db, eb, ec, aba, ede\}$.
The morphism γ, defined by

$$\gamma(a) = abaabbab, \quad \gamma(b) = aaabbbab, \quad \gamma(c) = aabbabab,$$
$$\gamma(d) = aabbbaba, \quad \gamma(e) = baaabbab,$$

maps this word w to a word with the desired properties.

Proof

Theorem (Proof of (1))

There exist infinitely long binary words avoiding both palindromes of length 6 and longer, and squares of words with length 3 and longer.

Theorem (Proof of (2))

There exist infinitely long ternary words avoiding both palindromes of length 3 and longer, and squares of words with length 2 and longer.

Proof

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There exist infinitely long binary words avoiding both palindromes of length 6 and longer, and squares of words with length 3 and longer.

Theorem (Proof of (2))

There exist infinitely long ternary words avoiding both palindromes of length 3 and longer, and squares of words with length 2 and longer.

The morphism ψ, that is defined by

$$\psi(a) = abbccaabccab, \quad \psi(b) = bccaabbcaabc, \quad \psi(c) = caabbccabbca,$$

maps all infinite square-free ternary words h to words with the desired properties.

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Theorem (Proof of (1))

There exist infinitely long binary words avoiding both palindromes of length 6 and longer, and squares of words with length 3 and longer.

Theorem (Proof of (2))

There exist infinitely long ternary words avoiding both palindromes of length 3 and longer, and squares of words with length 2 and longer.

Theorem (Proof of (3) - Pansiot)

There exist infinitely long words on a four letter alphabet that avoid $\frac{7}{5}$-powers (thus palindormes longer than 1 and squares).

Binary bounded case

Theorem

For any word $w \in \{a, b\}^+$ *and integer* $m \geq 0$, $\spadesuit^*_{(m,1)}(w)$ *is regular.*

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We express $\spadesuit^*_{(m,1)}(w)$ as a finite union and concatenation of several languages $\spadesuit^*_{(m,1)}(w')$ with w' being a word strictly shorter than w, and some other simple regular languages.

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We express $\spadesuit^*_{(m,1)}(w)$ as a finite union and concatenation of several languages $\spadesuit^*_{(m,1)}(w')$ with w' being a word strictly shorter than w, and some other simple regular languages.

A series of basic cases:

- words that have no maximal unary power greater than m,
- words of the form xy^qx
- words of the form xy^q or y^qx (x, y are letters)

Acknowledgement

Thank YOU for your attention

and the
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