Primitive Words are Unavoidable for Context-Free Languages

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Starting Question

Is the language Q of all primitive words context-free?

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Primitive: not an integer power.

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The language of all primitive words is NOT

- regular
- linear
- deterministic context-free
- unambiguous context-free

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- context-free **???** (Dömösi, Horváth, Ito, 1991)

Q is semi-linear and fulfills all known pumping conditions.

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Why does Q resist all attempts?

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Differences to standard pumping languages like $a^n b^n c^n$?

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There are very many primitive words.

What do I mean by *very many*?

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For two distinct primitive words p and q there are at most two non-primitive words in each of the languages p^*q and pq^*.

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[redacted]

Primitivity and Pumping

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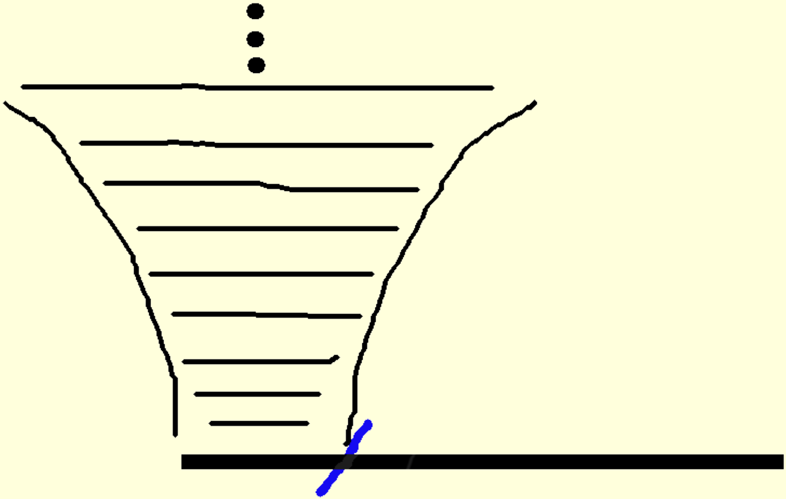

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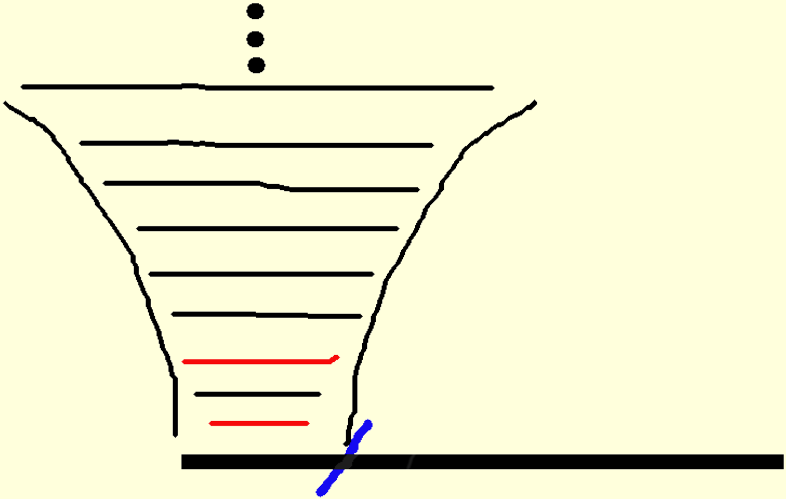

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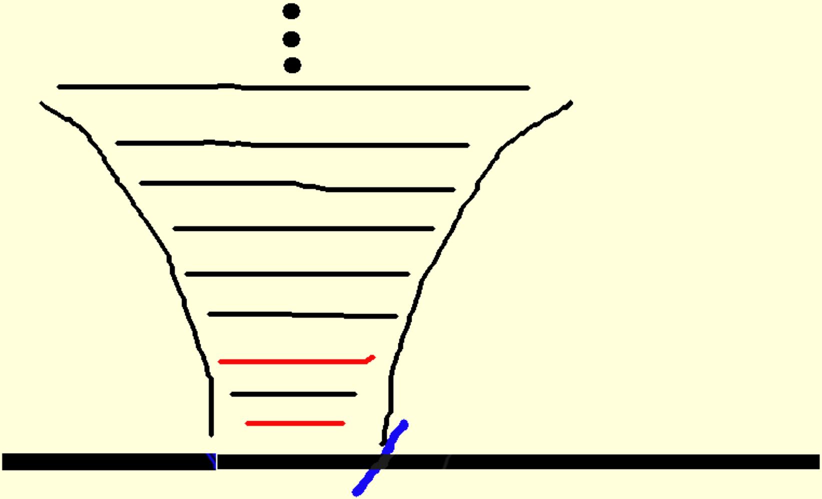
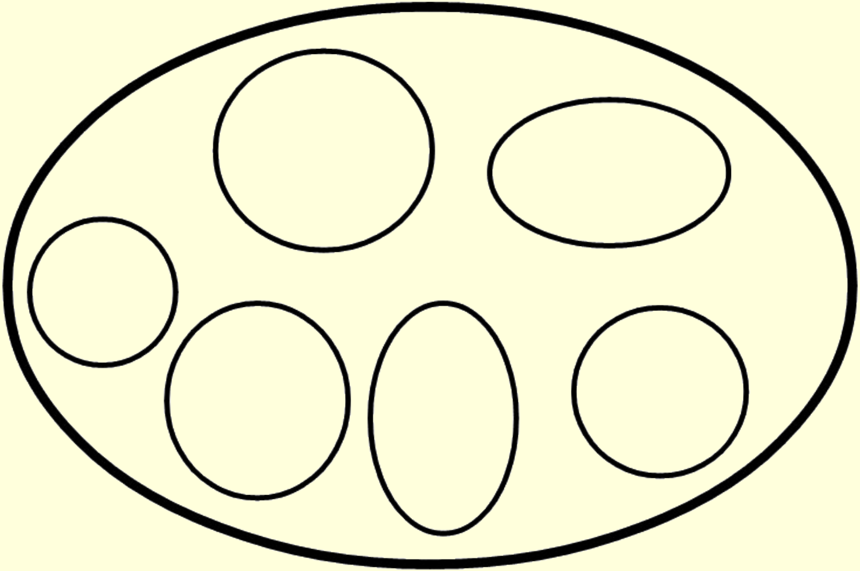


Unavoidability



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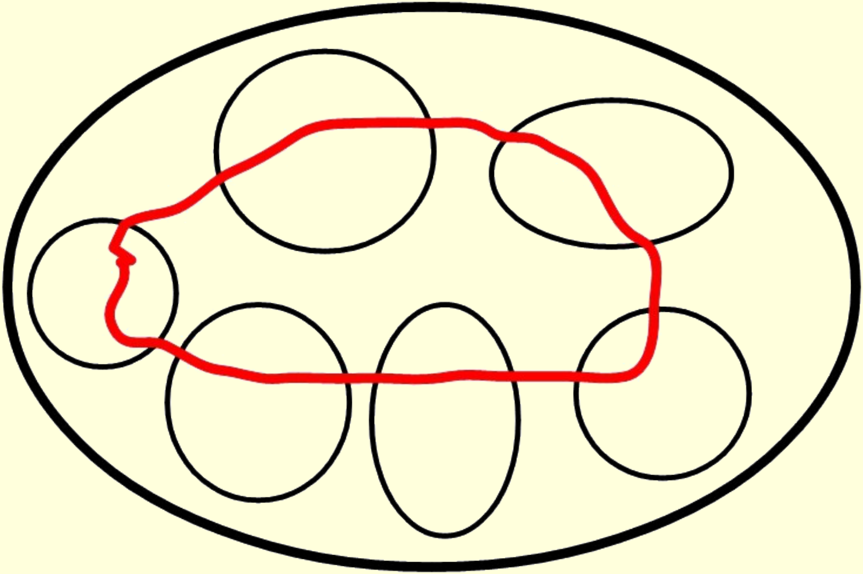

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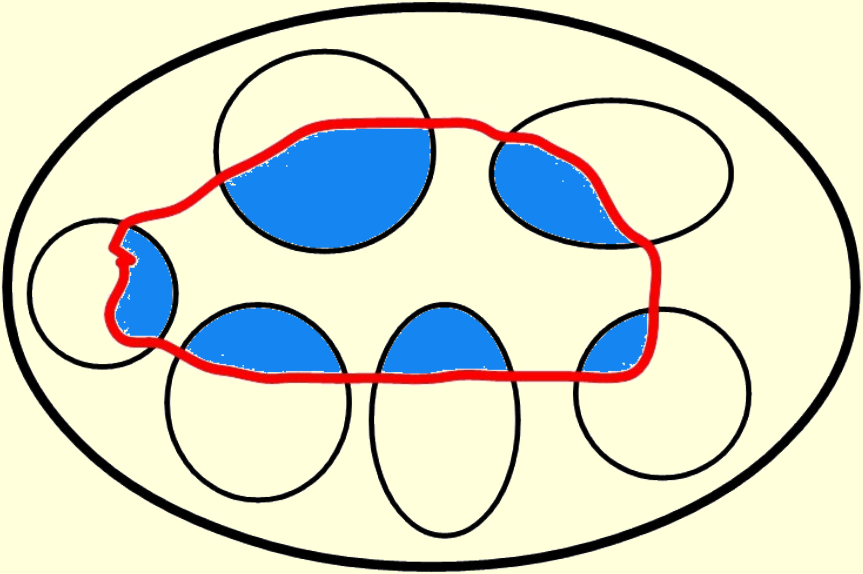
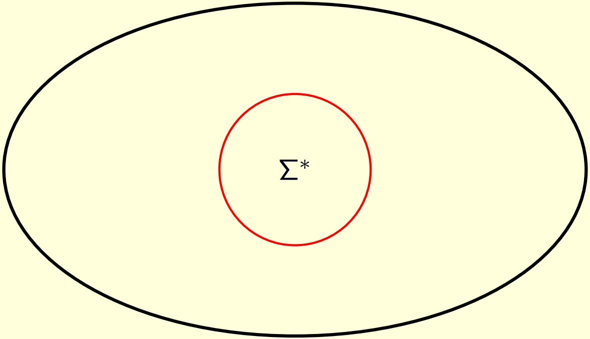


Trivial examples

Class



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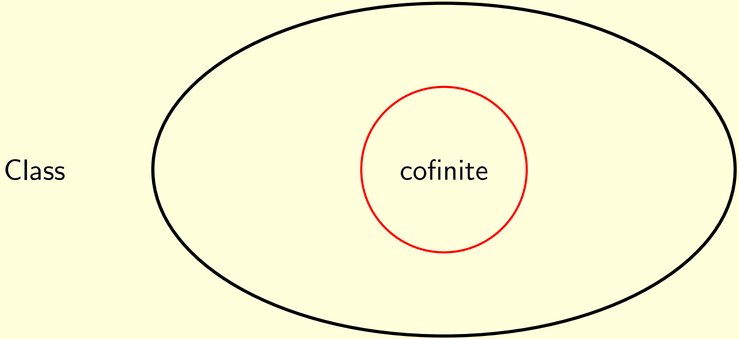
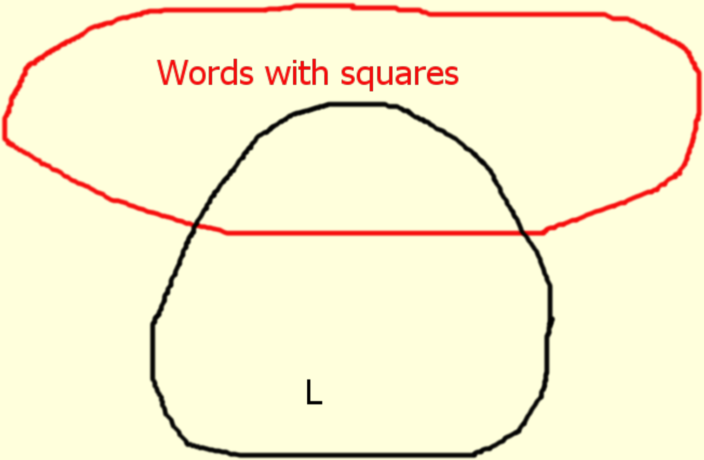


Example: Squares unavoidable in REG/CF



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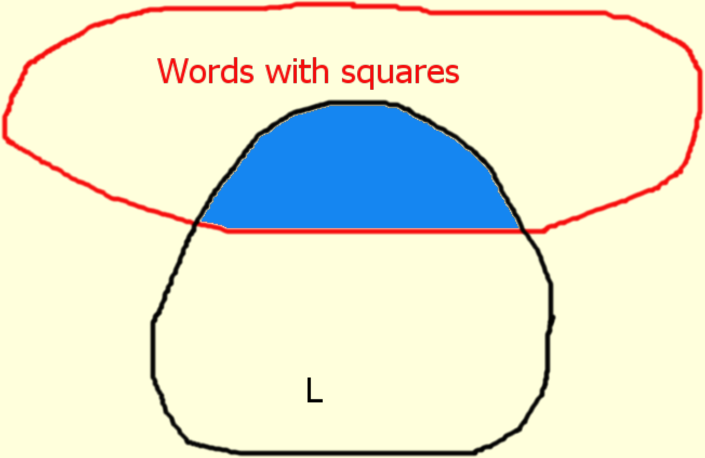

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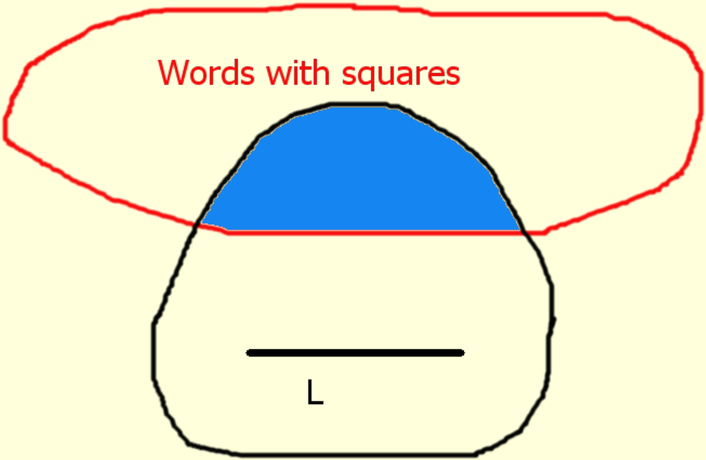

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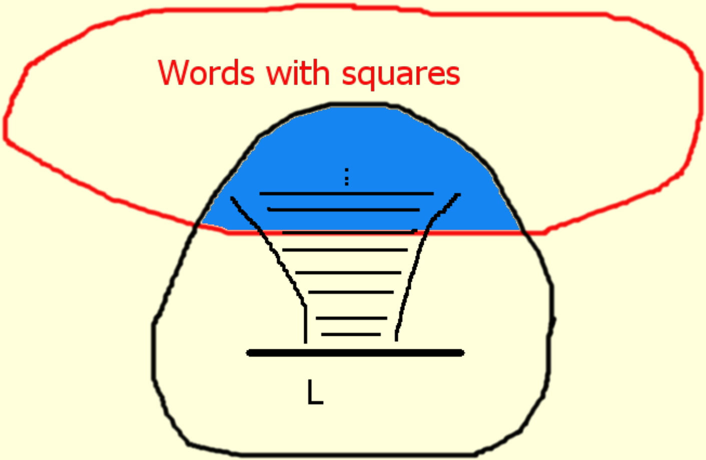

Unavoidable Languages

Theorem

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$(ab)^+$ has finite root, while ab^+ has infinite root (itself).

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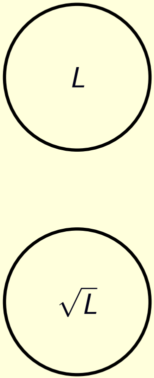

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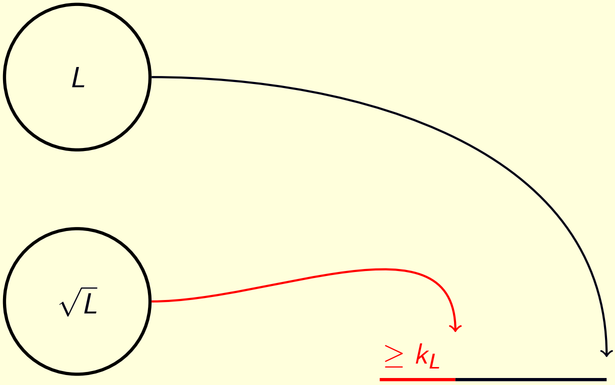

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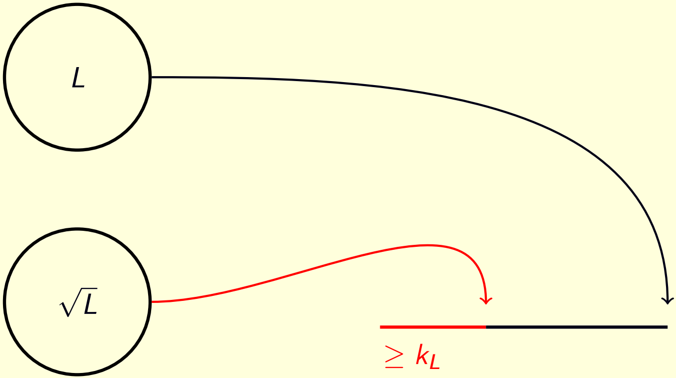

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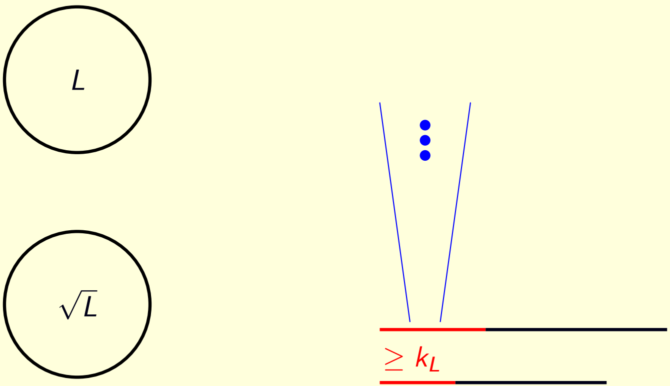
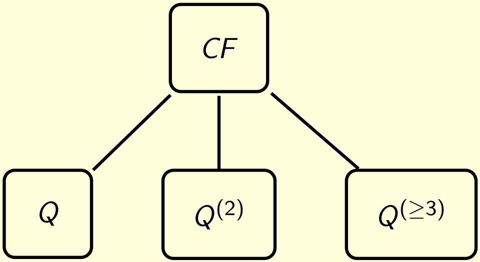


Primitive and non-primitive words in CF



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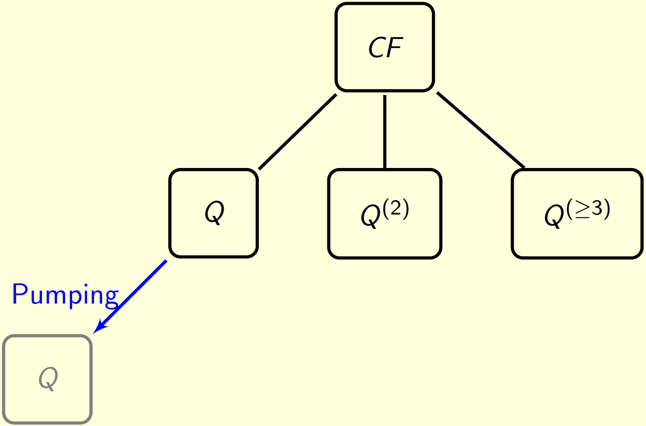

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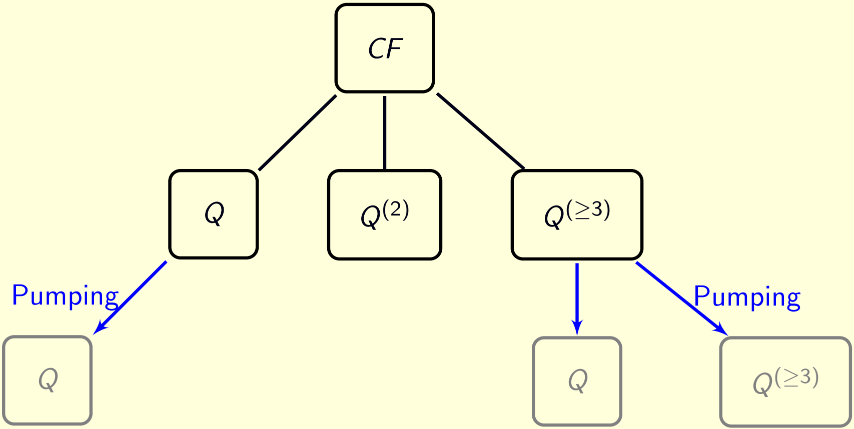

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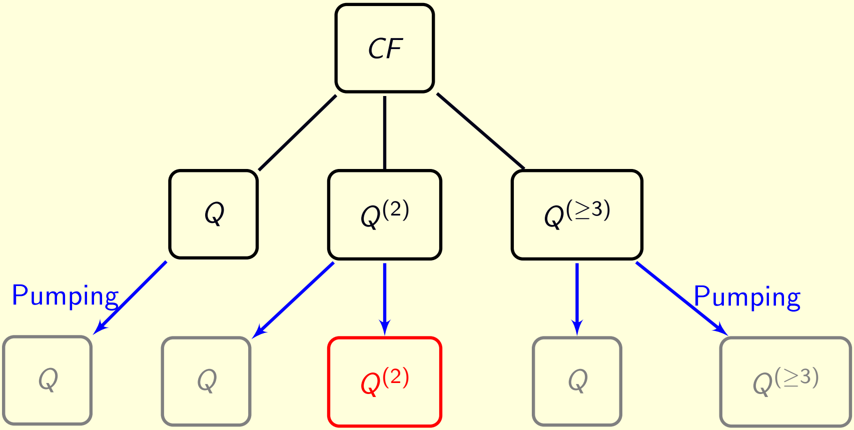

Non-Primitive Words in CF

Lemma

If $w_1 w_2^i w_3 w_4^i w_5 \in Q^{(2)}$ *for all* $i \geq 0$,

then w_2 *and* w_4 *are cyclic permutations of each other.*

Moreover, $w = (fg^k h)^2$ *such that*

$$w_1 w_2^i w_3 w_4^i w_5 = (fg^{i+k} h)^2.$$

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a b a c b b c a c b b a

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Conjugacy class

a	b	a	c	b	b	c	a	c	b	b	a							
	b	a	c	b	b	c	a	c	b	b	a	a						
		a	c	b	b	c	a	c	b	b	a	a	b					
			c	b	b	c	a	c	b	b	a	a	b	a				
				b	b	c	a	c	b	b	a	a	b	a	c			
					b	c	a	c	b	b	a	a	b	a	c	b		
						c	a	c	b	b	a	a	b	a	c	b	b	
							a	c	b	b	a	a	b	a	c	b	b	c

$\ddots$

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Consequence:

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All context-free subsets of $Q^{(2)}$ are finite unions of languages of the form

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For a context-free language it is decidable, whether it is a subset of $Q^{(2)}$.

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Suppose $L \cap Q$ finite

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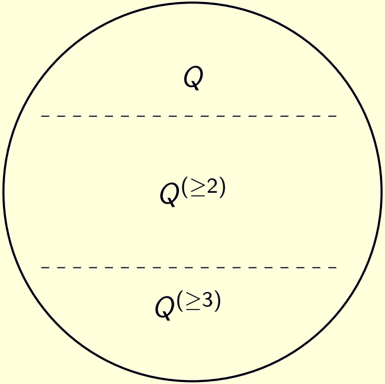

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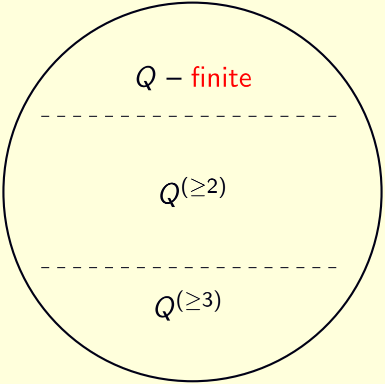

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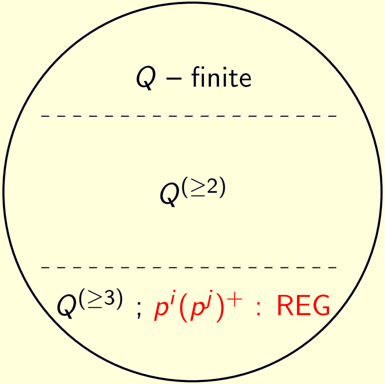

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Suppose $L \cap Q$ finite

Q – finite

$Q^{(\geq 2)}$; $L \setminus REG \setminus FIN$: LIN

$Q^{(\geq 3)}$; $p^i(p^j)^+$: REG

Primitive Words in CF

Corollary

Every context-free language that contains only finitely many primitive words is bounded.

Theorem

For a context-free language it is decidable, whether it contains infinitely many primitive words.

Primitive Words in CF

Is there some unavoidable subset of primitive words?

For example: palindromes.

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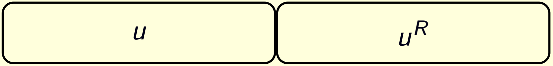
even:

$w \; w^R$

odd:

$w \; a \; w^R$

Two even palindromes in the same conjugacy class



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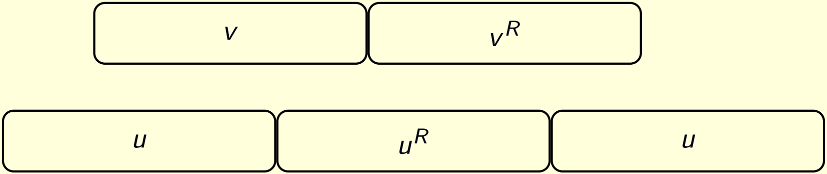

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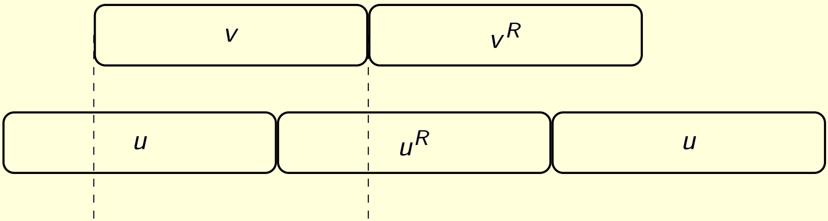

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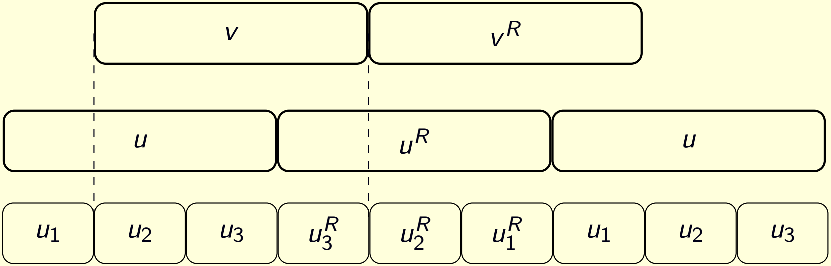

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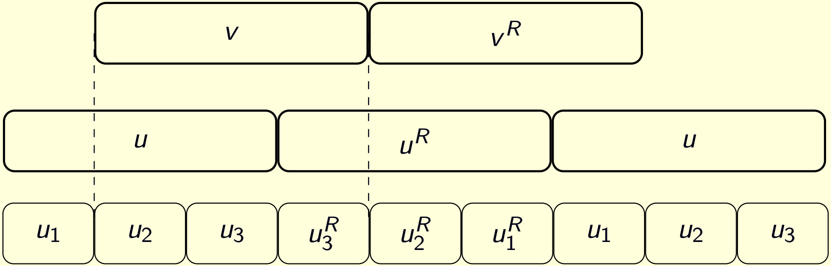

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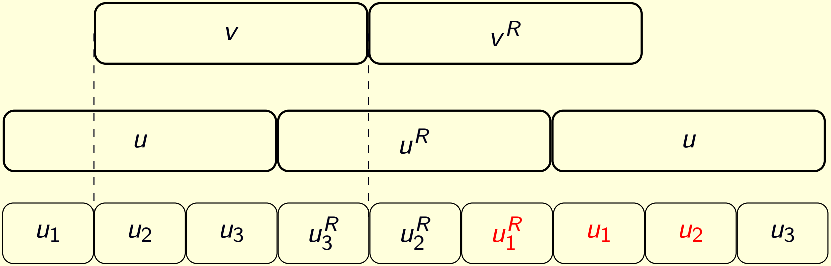

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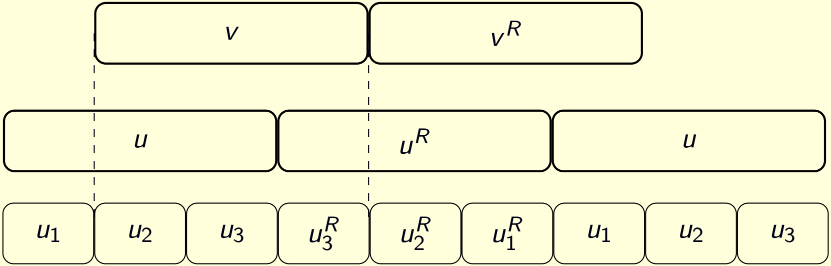
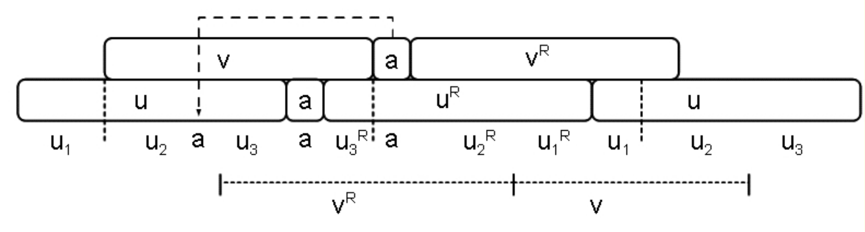


$$v = u_2 u_3 u_3^R = (u_2^R u_1^R u_1)^R \; = u_1^R u_1 u_2$$

vv^R conjugated to itself $\Longrightarrow$ not primitive

Two odd palindromes in the same conjugacy class



Primitive palindromes

Theorem (with Fazekas and Shikishima-Tsuji)

In a primitive word's conjugacy class there is either

- *no palindrome or*
- *one odd palindrome or*
- *two even palindromes.*

Much less primitive palindromes than primitive words.

Unavoidability of primitive palindromes

Theorem (with Fazekas and Shikishima-Tsuji)

The language $Q^{(2)}$ of squares of primitive words is strongly unavoidable for

- *non-regular*
- *context-free*
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- *containing only finitely many primitive words.*

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As a consequence, $Q \cup Q^{(2)}$ is strongly unavoidable for non-regular context-free palindromic languages.

Questions

Open Problem

Other nice examples of unavoidability?

Open Problem

Are non-primitive words unavoidable for non-regular and/or non-linear and/or non-deterministic context-free languages, or for context-free languages with infinite root?

Open Problem

Is the language Q of all primitive words context-free?