

Isomorphie von endlich präsentierten Strukturen

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Finitely presented structures

Goal: Infinite structures that have a finite representation.

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- ▶ computable structures
- ▶ $(\omega-)$ automatic structures, $(\omega-)$ tree automatic structures
- ▶ prefix recognizable graphs
- ▶ equational graphs
- ▶ pushdown graphs

Computable structures

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A (relational) structure $\mathcal{A} = (A, R_1, \dots, R_n)$ is **computable**, if

- ▶ $A \subseteq \mathbb{N}$ is a decidable set of naturals and
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Problem: Most interesting properties of computable structures are undecidable (e.g. has a computable graph at least one edge).

Automatic structures

Automatic structure (Büchi 1960, Khousseinov, Nerode 1995)

A structure $\mathcal{A} = (A, R_1, \dots, R_n)$ is **automatic**, if there exists a finite alphabet Σ such that:

- ▶ $A \subseteq \Sigma^*$ is a regular language and
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A synchronous 2-tape automaton working on a pair $(u, v) \in \Sigma^* \times \Sigma^*$:

v	b_0	b_1	b_2	\dots	b_{m-1}	b_m	$\#$	\dots	$\#$
u	a_0	a_1	a_2	\dots	a_{m-1}	a_m	a_{m+1}	\dots	a_n

Automatic structures

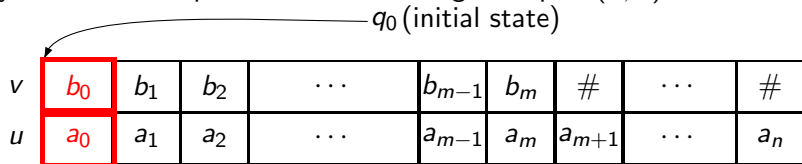
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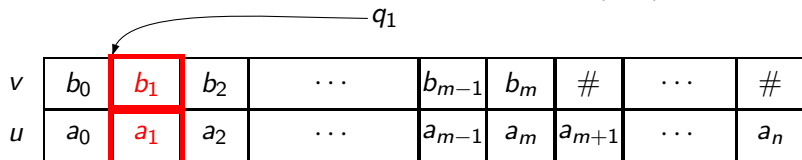
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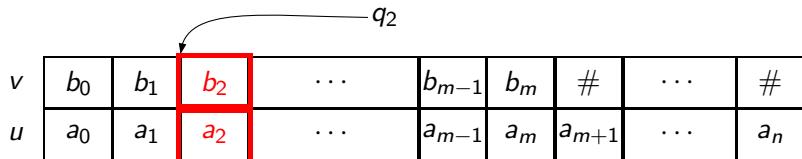
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				q_{m-1}					
v	b_0	b_1	b_2	\dots	b_{m-1}	b_m	$\#$	\dots	$\#$
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	b_0	b_1	b_2	\dots	b_{m-1}	b_m	$\#$	\dots	$\#$
v									
	a_0	a_1	a_2	\dots	a_{m-1}	a_m	a_{m+1}	\dots	a_n
u									

q_m →

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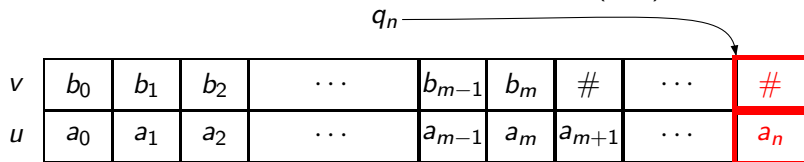
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 q_{n+1} (final state?) 

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- ▶ Configuration graphs of Turing machines

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- ▶ A field is automatic if and only if it is finite (Khoussainov, Nies, Rubin, Stephan 2007).

Isomorphism

The **isomorphism problem** for a class \mathcal{C} of finitely presented structures (e.g., computable structures, automatic structures):

INPUT: two structures $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{C}$

QUESTION: $\mathcal{A}_1 \cong \mathcal{A}_2$ (are \mathcal{A}_1 and \mathcal{A}_2 isomorphic)?

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To get more interesting results, we need a bit of recursion theory.

Arithmetical hierarchy

Arithmetical hierarchy (Kleene 1943)

A set $A \subseteq \mathbb{N}$ belongs to the class Σ_n^0 (resp. Π_n^0) if there exists a quantifier-free arithmetical formula $\varphi(x, \bar{y}_1, \dots, \bar{y}_n)$ (with $+$ and \times) such that

$$A = \{x \in \mathbb{N} \mid \exists \bar{y}_1 \forall \bar{y}_2 \exists \bar{y}_3 \cdots \forall / \exists \bar{y}_n : \varphi(x, \bar{y}_1, \dots, \bar{y}_n)\}$$

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$$\text{REC} = \Sigma_1^0 \cap \Pi_1^0 \quad \Sigma_2^0 \cap \Pi_2^0 \quad \Sigma_3^0 \cap \Pi_3^0 \quad \Sigma_4^0 \cap \Pi_4^0 \quad \dots$$

Analytical hierarchy

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A set $A \subseteq \mathbb{N}$ belongs to the class Σ_n^1 (resp. Π_n^1) if there exists a first-order arithmetical formula $\varphi(x, X_1, \dots, X_n)$ (with $+$ and \times and unary predicates X_1, \dots, X_n) such that

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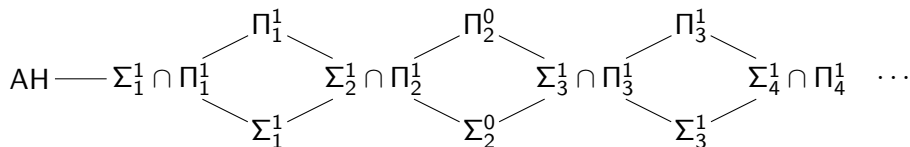
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Intuitive meaning: In order to express that $\mathcal{A}_1 \cong \mathcal{A}_2$ for computable structures $\mathcal{A}_1, \mathcal{A}_2$, there is no better way than saying:

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Theorem 4 (Rubin 2004)

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Theorem 5 (Khousainov, Nies, Rubin, Stephan 2007)

The isomorphism problem is Σ_1^1 -complete for *automatic structures* (even *automatic successor trees*).

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- ▶ *automatic Boolean algebras*.

Theorem 4 (Rubin 2004)

The isomorphism problem for *automatic graphs of bounded degree* is Π_3^0 -complete.

Theorem 5 (Khousainov, Nies, Rubin, Stephan 2007)

The isomorphism problem is Σ_1^1 -complete for *automatic structures* (even *automatic successor trees*).

Basic ideas: Configuration graphs of Turing machines

Isomorphism problems for equivalence structures

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Theorem 6 (Kuske, Liu, L 2010)

*The isomorphism problem for **automatic equivalence structures** is Π_1^0 -complete.*

Theorem 7 (Calvert, Cenzer, Harizanov, Morozov 2005)

*The isomorphism problem for **computable equivalence structures** is Π_4^0 -complete.*

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Then, $\mathcal{E}_1 \cong \mathcal{E}_2$ if and only if **for all** $n \in \mathbb{N} \cup \{\infty\}$:

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For a fixed $n \in \mathbb{N} \cup \{\infty\}$, the number of \mathcal{E}_i -equivalence classes of size n can be computed effectively.

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Theorem 8 (Matiyasevich 1971)

For every Σ_1^0 -set $X \subseteq \mathbb{N}$ there is a polynomial $p \in \mathbb{Z}[x_0, \dots, x_k]$ (which can be computed effectively from an index for X) with:

$$X = \{n \in \mathbb{N} \mid \exists a_1, \dots, a_k \in \mathbb{N} : p(n, a_1, \dots, a_k) = 0\}.$$

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Corollary

The following problem is Π_1^0 -complete:

INPUT: polynomials $p_1(x_1, \dots, x_k), p_2(x_1, \dots, x_k) \in \mathbb{N}[x_1, \dots, x_k]$

QUESTION: $\forall x_1, \dots, x_k \in \mathbb{N} : p_1(x_1, \dots, x_k) \neq p_2(x_1, \dots, x_k)$?

Coding polynomials by automata

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Step 1: From $p(x_1, \dots, x_k)$ we can construct inductively a nondeterministic finite automaton (NFA) \mathcal{A} over the alphabet $\{a_1, \dots, a_k\}$ such that $L(\mathcal{A}) = a_1^* a_2^* \cdots a_k^*$ and

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Step 2: We define the **run automaton** $\text{Run}(\mathcal{A})$ as the NFA (over the alphabet Δ) that results from \mathcal{A} by replacing in \mathcal{A} every transition

$$q \xrightarrow{a} q' \quad \text{by the transition} \quad q \xrightarrow{(q,a,q')} q'.$$

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Then we have for all $n \in \mathbb{N}$:

$$\begin{aligned} \mathcal{E}(p) \text{ has an equivalence class of size } n & \\ \iff & \\ \exists w \in a_1^* a_2^* \cdots a_k^* : n = \#[\text{accepting runs of } \mathcal{A} \text{ on input } w] & \\ \iff & \\ n \in \text{Img}(p) & \end{aligned}$$

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We construct two automatic equivalence structures \mathcal{E}_1 and \mathcal{E}_2 such that $\mathcal{E}_1 \cong \mathcal{E}_2$ if and only if

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Let $C(x, y) = (x + y)^2 + 3x + y$ (injective from $\mathbb{N} \times \mathbb{N}$ to \mathbb{N}) and

$$S_1(x_1, \dots, x_k) = C(p_1(x_1, \dots, x_k), p_2(x_1, \dots, x_k))$$

$$S_2(x_1, x_2) = C(x_1 + x_2 + 1, x_1)$$

$$S_3(x_1, x_2) = C(x_1, x_1 + x_2 + 1).$$

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Final step

For an equivalence structure $\mathcal{E} = (A, \equiv)$ let $\aleph_0 \cdot \mathcal{E}$ be the equivalence structure

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The following two observations conclude the proof:

- ▶ If \mathcal{E} is automatic, then $\aleph_0 \cdot \mathcal{E}$ is automatic too.
- ▶ For two countable equivalence structures \mathcal{E}_1 and \mathcal{E}_2 we have:

$$\mathcal{E}_1 \sim \mathcal{E}_2 \iff \aleph_0 \cdot \mathcal{E}_1 \cong \aleph_0 \cdot \mathcal{E}_2.$$

Isomorphism problem for automatic structures

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- ▶ *The isomorphism problem for automatic order trees is Σ_1^1 -complete.*
- ▶ *The isomorphism problem for automatic linear orders is Σ_1^1 -complete.*
- ▶ *The isomorphism problem for scattered automatic linear orders can be reduced to $\text{FOTh}(\mathbb{N}, +, \times)$ (but decidability is open).*

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Theorem 12 (Huschenbett, Kartzow, Liu, L 2012)

The isomorphism problem for well-founded tree-automatic order trees is complete for the hyperarithmetical level $\Delta_{\omega\omega}^0$.

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Theorem 14 (Finkel 2010)

There exist two ω -tree automatic structures \mathcal{A}_1 and \mathcal{A}_2 and two models \mathcal{S}_1 and \mathcal{S}_2 of ZFC such that

- ▶ $\mathcal{S}_1 \models \mathcal{A}_1 \cong \mathcal{A}_2$
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Open problem: Isomorphism problem for prefix recognizable graphs.

Regular trees and regular linear orders

Theorem 16 (L, Mathissen 2011)

It is P-complete to check for two given DFAs A_1, A_2 , whether $(L(A_1), \leq_{pref}) \cong (L(A_2), \leq_{pref})$ (resp. $(L(A_1), \leq_{lex}) \cong (L(A_2), \leq_{lex})$).

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Context-free languages

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It is Σ_1^1 -complete to check for two given deterministic pushdown automata (even visibly pushdown automata) A_1, A_2 , whether $(L(A_1), \leq_{pref}) \cong (L(A_2), \leq_{pref})$ (resp. $(L(A_1), \leq_{lex}) \cong (L(A_2), \leq_{lex})$)

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Theorem 20 (Kuske 2013)

It is undecidable to check for deterministic real-time 1-counter automata A_1, A_2 with $L(A_1) \cap L(A_2) = \emptyset$, whether $(L(A_1) \cup L(A_2), \leq_{lex}) \cong (\mathbb{Q}, \leq)$.

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