Chapter 1

Basic definitions

1.1 Order theoretic definitions

1.1.1 Well quasi orders

Let A be a set. A *quasi order* on A is a binary relation $\preceq \subseteq A \times A$ that is transitive and reflexive. The tuple $(A, \preceq)$ is called *quasi ordered set*. So let $(A, \preceq)$ be a quasi ordered set, $a \in A$ and $X \subseteq A$. Then we define $\uparrow a := \{b \in A \mid a \preceq b\}$ and $\uparrow X := \bigcup_{x \in X} \uparrow x$. A set $B \subseteq A$ is a *basis of* X if $\uparrow B = \uparrow X$. Note that any set X has a basis, namely itself or $\uparrow X$. In the literature, one often defines a basis for sets X with $X = \uparrow X$, only, but in our context, it is more convenient to extend the classical definition slightly.

We call a sequence $(a_i)_{i \in \mathbb{N}}$ in a quasi ordered set $(A, \preceq)$ *good* if there are $i < j$ with $a_i \preceq a_j$. If no such indices exist, the sequence is *bad*. A *well quasi order* is a quasi order $\preceq$ on a set A where any sequence in A is good. A *wqo* is a quasi ordered set $(A, \preceq)$ where $\preceq$ is a well quasi order. Occasionally, we use wqo as an abbreviation of well quasi order, too.

In a wqo, any set contains a finite basis: Let $(A, \preceq)$ be a wqo and $a_i \in A$ for $i \in \mathbb{N}$. Let M consist of all indices $i \in \mathbb{N}$ such that $x_i \not\preceq x_j$ for any $j > i$. Since $(A, \preceq)$ is a wqo, this set is finite. Choose $i_0 \in \mathbb{N}$ with $M \leq i_0$. Then, inductively, we find $i_{n+1} > i_n$ with $a_{i_n} \preceq a_{i_{n+1}}$, i.e. the sequence $(a_i)_{i \in \mathbb{N}}$ contains an infinite non-decreasing subsequence. Now let $X \subseteq A$. An element $x \in X$ is *minimal in* X if for any $y \in X$ with $y \preceq x$ we get $x \preceq y$. By $\min(X)$, we denote the set of minimal elements of X. Let $\sim = \preceq \cap \succeq$. Since $\preceq$ is transitive and reflexive, $\sim$ is an equivalence relation. Note that $\uparrow x = \uparrow y$ for $x, y \in A$ whenever $x \sim y$. Let $(x_i)_{i \in \alpha}$ be an enumeration of $\min(X)$ for some ordinal α. Furthermore, let $i_0 = 0$. Inductively, let $n \in \mathbb{N}$ and assume that $i_n \in \alpha$ is chosen. If there exists $i > i_n$ such that $x_i \not\sim x_{i_j}$ for $0 \leq j \leq n$, let i_{n+1} be the minimal such i. If this construction does not terminate, we get a sequence $(x_{i_n})_{n \in \mathbb{N}}$ with $x_{i_n} \not\sim x_{i_m}$ for $n < m$. Since $(A, \preceq)$ is a wqo, there is $n < m$ with $x_{i_n} \preceq x_{i_m}$. Since $x_{i_m} \in \min(X)$, this implies $x_{i_m} \preceq x_{i_n}$, contradicting the choice of i_m. Thus, there is $k \in \mathbb{N}$ such that we find for $x \in \min(X)$ an index $0 \leq j \leq k$ with $x \sim x_{i_j}$. Now let $y \in X$. Then there exists $x \in \min(X)$ with $x \preceq y$ for otherwise we found an infinite sequence $(y_i)_{i\in\mathbb{N}}$ with $y_i \succ y_{i+1}$, i.e. in particular with $y_i \not\preceq y_j$ for $i < j$. Thus, the set $\{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\}$ is a finite basis of X, i.e. we showed that any set $X \subseteq A$ contains a finite basis.

Next, we want to define from a wqo $(A, \preceq)$ a quasi order on the set of finite words over A. So let $v = a_1a_2\ldots a_n$ and $w = b_1b_2\ldots b_m$ be words over A. We define $v \preceq^\star w$ iff there exists a sequence $0 < i_1 < i_2 < \cdots < i_n \leq m$ such that $a_j \preceq b_{i_j}$ for $1 \leq j \leq n$, i.e. if v is dominated by some subword of w letter by letter. Clearly, $\preceq^\star$ is transitive and reflexive, i.e. it is a quasi order on the set of words $A^\star$ over A.

Higman's Theorem [Hig52] $(A^\star, \preceq^\star)$ is a wqo.

Proof.[1] By contradiction suppose $\preceq^\star$ is no wqo. Then there exists a bad sequence in $A^\star$. Let v_0 be a word of minimal length such that there exists a bad sequence $(w_i)_{i\in\mathbb{N}}$ with $w_0 = v_0$. Inductively, assume we found $v_0, v_1, \ldots, v_n \in A^\star$ such that there exists a bad sequence starting with these words. Then let $v_{n+1} \in A^\star$ be a word of minimal length such that $v_0, v_1, \ldots, v_n, v_{n+1}$ can be extended to a bad sequence. Note that in particular $v_i \not\preceq^\star v_{n+1}$ for $0 \leq i \leq n$. This construction results in a bad sequence $(v_i)_{i\in\mathbb{N}}$ such that, for any $i \in \mathbb{N}$ and word $w \in A^\star$ shorter than v_i, the tuple $v_0, v_1, \ldots, v_{i-1}, w$ cannot be extended to a bad sequence. Since the empty word is dominated by any word, in addition none of these words is empty. For $i \in \mathbb{N}$, let $a_i \in A$ be the first letter of v_i and let w_i be the remaining word, i.e. $a_iw_i = v_i$. Since $(A, \preceq)$ is a wqo, the sequence $(a_i)_{i\in\mathbb{N}}$ contains an infinite non-decreasing subsequence $(a_{i_j})_{j\in\mathbb{N}}$. Now consider the sequence

$$v_0, v_1, \ldots, v_{i_0-1}, w_{i_0}, w_{i_1}, w_{i_2}, \ldots$$

in $A^\star$. For $1 \leq i < j < i_0$, we have $v_i \not\preceq^\star v_j$ since the words v_n form a bad sequence. For $1 \leq i < i_0$ and $j \in \mathbb{N}$, we get $v_i \not\preceq^\star w_{i_j}$ for otherwise $v_i \preceq^\star a_{i_j}w_{i_j} = v_{i_j}$, contradicting that the words v_n form a bad sequence. Now let $i < j$ and assume $w_{i_i} \preceq^\star w_{i_j}$. Since $a_{i_i} \preceq a_{i_j}$, this implies $v_{i_i} = a_{i_i}w_{i_i} \preceq^\star a_{i_j}w_{i_j} = v_{i_j}$, again a contradiction. Hence the sequence above is bad. But this contradicts the fact that v_{i_0} is properly longer than w_{i_0} and that by our choice of v_{i_0}, the tuple $v_0, v_1, \ldots, v_{i_0-1}, w_{i_0}$ cannot be extended to a bad sequence. Thus, indeed, $\preceq^\star$ is a wqo on the set of finite words over A. □

[1] This proof of Higman's theorem follows a proof given in [Die96] where the idea of a minimal bad subsequence is attributed to Nash-Williams [NW63].

1.1.2 Partial orders

Let A be a set. A quasi order $\leq$ on A is a *(partial) order* if it is antisymmetric. Then $(A, \leq)$ is a *partially ordered set* or *poset* for short. Two elements $a, b \in A$ are *incomparable* (denoted $a \parallel b$) if neither $a \leq b$ nor $b \leq a$. By $\lessgtr$, we denote the union of $<$ and $>$. Hence $a \not\lessgtr b$ iff $a \parallel b$ or $a = b$. An element $c \in A$ *covers* $a \in A$ iff $a < c$ and if $a < b \leq c$ implies $b = c$. We write $a \prec c$ whenever a is covered by c.

The set A is an *antichain* if any two distinct elements of A are incomparable. If, on the contrary, any two of its elements are comparable (i.e. not incomparable), then A is *linearly ordered* or a *chain*. An (anti-)chain X in $(A, \leq)$ is a subset $X \subseteq A$ such that $(X, \leq \cap X \times X)$ is an (anti-)chain. The set $X \subseteq A$ is *convex* if for any $x \leq y \leq z$ with $x, z \in X$ the element y belongs to X, too. A subset X of A is a *filter* if $x \in X$ and $x \leq y$ imply $y \in X$. Dually, a set $X \subseteq A$ is an *ideal* if $x \in X$ and $x \geq y$ imply $y \in X$. Since traditionally ideals were called heriditary sets, the set of ideals of $(A, \leq)$ is denoted by $\mathbb{H}(A, \leq)$.

Recall that $\uparrow a = \{b \in A \mid a \leq b\}$. We call this set the *principal filter generated by* a. Dually, $\downarrow a = \{b \in A \mid a \geq b\}$ is the *principal ideal generated by* a. By $\updownarrow a$, we denote the union of $\uparrow a$ and $\downarrow a$, i.e. the set of elements of A that are comparable with a. The intersection of $\uparrow a$ and $\downarrow b$ is denoted by $[a, b]$. It is the interval with endpoints a and b. Note that this interval is nonempty iff $a \leq b$. For $X \subseteq A$, let $\uparrow X := \bigcup_{x \in X} \uparrow x$ and dually $\downarrow X := \bigcup_{x \in X} \downarrow x$ denote the generated filter and ideal, respectively. An ideal I is finitely generated if there exists a finite set X such that $I = \downarrow X$. The set of finitely generated ideals will be denoted by $\mathbb{H}_f(A, \leq)$.

For $X \subseteq A$ and $a \in A$, we write $X \leq a$ whenever $x \leq a$ for all $x \in X$. In this case a is an *upper bound* of X. It is a *minimal upper bound* if $X \leq x \leq a$ implies $x = a$. By $\mathrm{mub}(X)$, we denote the set of minimal upper bounds of X. An upper bound a of X that is dominated by any upper bound of X is the *least upper bound*, *supremum* or *join* of X. It is denoted by $\sup(X)$ or $\bigvee X$. The supremum of a two-element set $\{a, b\}$ is denoted by $a \vee b$. Dually, *lower bound*, *maximal lower bound*, *largest lower bound* or *infimum* or *meet* are defined. The infimum of $X \subseteq A$ is denoted by $\inf(X)$, $\bigwedge(X)$ or $a \wedge b$ if $X = \{a, b\}$. An element $a \in A$ is *join-irreducible* if $x \vee y = a$ implies $a \in \{x, y\}$ and $a \not\leq A$. By $\mathbb{J}(A, \leq)$, we denote the set of join-irreducible elements of A.

Let $(A, \leq)$ be a poset and $a \in A$. The *width* $w(A, \leq)$ *of* $(A, \leq)$ is the supremum of the sizes of all antichains in A. The *height of* a is the supremum of all sizes of chains $C < a$. We denote the height of a in $(A, \leq)$ by $h(a, (A, \leq))$ or shorter by $h(a, A)$ or just by $h(a)$. Note that the minimal elements of a poset have height 0. The *length of* $(A, \leq)$ is the supremum of the heights of the elements of A.

A partially ordered set $(A, \leq)$ is a *join-semilattice* iff any finite subset of A has a supremum. It is a *lattice* if in addition any finite subset of A has an infimum. Note that if $(A, \leq)$ is a lattice so is $(A, \geq)$. Two intervals $[a, b]$ and $[a', b']$ in a lattice $(A, \leq)$ are *transposed* iff $a = b \wedge a'$ and $b' = b \vee a'$.

A lattice of finite length is *semimodular* if $a \wedge b \longrightarrow\!\!< a$ implies $b \longrightarrow\!\!< a \vee b$. A lattice $(A, \leq)$ is *modular* if $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$. A lattice $(A, \leq)$ of finite length is modular iff both $(A, \leq)$ and $(A, \geq)$ are semimodular. Furthermore, in a modular lattice $(A, \leq)$, transposed intervals are isomorphic. More precisely, let $[b \wedge c, b]$ and $[c, b \vee c]$ be two transposed intervals and define $f(x) := x \vee c$ for $b \wedge c \leq x \leq b$. Then this mapping f is an isomorphism of the two intervals [Bir73, Theorem I.7.13]. A lattice $(A, \leq)$ is *distributive* if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for any $a, b, c \in A$. Then one also has the dual identity $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. Furthermore, any distributive lattice is modular.

Let $(A, \leq)$ be a poset. Then the set of ideals $X = \mathbb{H}(A, \leq)$ can be ordered by inclusion. The poset $(X, \subseteq)$ is a lattice, the supremum is given by union and the infimum by intersection. One can easily check that it is even a distributive lattice and that an ideal $I \in \mathbb{H}(A, \leq)$ is join-irreducible in this lattice if it is a principal ideal. Note that an ideal $I \in \mathbb{H}(A, \leq)$ is join-irreducible iff it covers a unique element of $(\mathbb{H}(A, \leq), \subseteq)$.

Now let $(L, \leq)$ be a distributive lattice. Then $(A, \leq) := (\mathbb{J}(L, \leq), \leq)$ is a poset and $(\mathbb{H}(A, \leq), \subseteq)$ is a distributive lattice. If L is finite, this latter lattice is isomorphic to $(L, \leq)$ [Bir73, Theorem I.4.3].

1.2 Monoid theoretic definitions

A monoid is a set M equipped with a binary operation $\cdot : M^2 \to M$ that is associative and admits a neutral element 1. The *left divisibility relation* on a monoid $(M, \cdot, 1)$ is defined by $x \leq z$ iff there exists $y \in M$ with $x \cdot y = z$. Since the multiplication $\cdot$ is associative, this relation is transitive. It is in addition reflexive since a monoid contains a neutral element. Hence $(M, \leq)$ is a quasi ordered set. Since $1 \leq M$, the set $\{1\}$ is a basis of $(M, \leq)$. In general, $\leq$ is neither a partial order relation since it need not be antisymmetric (consider the reals with addition) nor a wqo (consider the nonnegative reals with addition).

An alphabet Σ is a nonempty finite set. The set $\Sigma^\star$ of words over Σ gets a monoid structure when equipped with the usual concatenation of words. The neutral element is the empty word, which is denoted by ε. The monoid $(\Sigma^\star, \cdot, \varepsilon)$ is called the *free monoid over* Σ.

Let $(M_i, \cdot_i, 1_i)$ be monoids for $i = 1, 2$ and let $f : M_1 \to M_2$ be a function. This function is a *homomorphism* if $f(x \cdot_1 y) = f(x) \cdot_2 f(y)$ for $x, y \in M_1$ and $f(1_1) = f(1_2)$. A *congruence* on the monoid M_1 is an equivalence relation $\sim$ such that $x_i \sim y_i$ for $i = 1, 2$ and $x_i, y_i \in M_1$ imply $x_1 \cdot_1 x_2 \sim y_1 \cdot_1 y_2$.

A *dependence alphabet* or *trace alphabet* is an alphabet Σ endowed with a binary relation D that is reflexive and symmetric. The relation D is called *dependence relation* and its complement $I = \Sigma^2 \setminus D$ is the *independence relation*. From a dependence alphabet (Σ, D), one defines the trace monoid $\mathbb{M}(\Sigma, D)$ as follows: First, let $\sim$ denote the least congruence on the free monoid $(\Sigma^\star, \cdot, \varepsilon)$

with $ab \sim ba$ for $(a, b) \in I$. Note that two equivalent words $v \sim w$ over Σ have the same length. Then $\mathbb{M}(\Sigma, D) = \Sigma^\star/\sim$ is a monoid whose elements are called *traces*. Thus, traces are equivalence classes of words. The length $|x|$ of a trace is the length of any of its representatives. Originally, these monoids where considered by Cartier & Foata [CF69] under the name free partially commutative monoids. The name *trace monoid* was coined by Mazurkiewicz [Maz77].

Besides this algebraic definition of trace monoids, there is another, equivalent, construction of them: Again, one starts with a dependence alphabet (Σ, D). A *dependence graph* is either empty or a triple $(V, \preceq, \lambda)$ where $(V, \preceq)$ is a finite poset and $\lambda : V \to \Sigma$ is a mapping such that for $x, y \in V$, one has

- $x \parallel y$ implies $(\lambda(x), \lambda(y)) \notin D$ and
- $x \longrightarrow\!\!\!< y$ implies $(\lambda(x), \lambda(y)) \in D$.

As usual in mathematics, isomorphic dependence graphs are not differentiated. On the set of dependence graphs one defines a binary operation $\cdot$ by

$$(V_1, \preceq_1, \lambda_1) \cdot (V_2, \preceq_2, \lambda_2) = (V_1 \dot\cup V_2, \preceq_1 \cup \preceq_2 \cup (\preceq_1 \circ E \circ \preceq_2), \lambda_1 \cup \lambda_2)$$

where $E = \{(x, y) \in V_1 \times V_2 \mid (\lambda_1(x), \lambda_2(y)) \in D\}$. Then one can easily check that this operation is associative and that the empty dependence graph is a neutral element.

For $a \in \Sigma$, let $t_a = (\{a\}, \{(a, a)\}, \{(a, a)\})$ denote the dependence graph with one vertex that is labeled by the letter a. Since the monoid $\mathbb{M}(\Sigma, D)$ is generated by the elements $[a]$ for $a \in \Sigma$, the mapping $[a] \mapsto t_a$ can uniquely be extended to a homomorphism from the trace monoid $\mathbb{M}(\Sigma, D)$ to the monoid of dependence graphs. It turns out that this homomorphism is an isomorphism of the monoids. Hence traces can be considered as labeled partially ordered sets. The relation between traces, i.e. equivalence classes of words, and labeled posets can be seen in another light, too:

Recall that $x \leq z$ iff there exists $y \in \mathbb{M}(\Sigma, D)$ such that $x \cdot y = z$. Since $x < z$ implies $|x| < |y|$, on the trace monoid $\mathbb{M}(\Sigma, D)$, the left divisibility relation is a partial order. One can show that $(\downarrow x, \leq)$ is a distributive lattice for any trace x. Let $(V, \leq, \lambda)$ be the dependence graph associated to x. Then the partial order of join-irreducibles of $(\downarrow x, \leq)$ is isomorphic to $(V, \leq)$. Vice versa, $(\downarrow x, \leq)$ is isomorphic to the set of ideals of $(V, \leq)$, i.e. to $(\mathbb{H}(V, \leq), \subseteq)$.

1.3 Logic

In this work, a *graph* is a finite set V together with a binary relation E, i.e., we consider directed graphs without multiple edges. A *dag* is a directed acyclic graph. A Σ-*labeled graph* is a graph (V, E) together with a mapping $\lambda : V \to \Sigma$, i.e., we consider vertex-labeled graphs.

Next, we introduce the monadic second order logic MSO that allows to reason on Σ-labeled graphs: So let Σ be an alphabet, i.e. a finite set. Let $V_e = \{x_i \mid i \in \mathbb{N}\}$ be a countable set of *elementary variables* and $V_s = \{X_i \mid i \in \mathbb{N}\}$ a countable set of *set variables*. There are three kinds of *atomic formulae*, namely $E(x_i, x_j)$, $X_j(x_i)$ and $\lambda(x_i) = a$ for $i, j \in \mathbb{N}$ and $a \in \Sigma$. *Formulas* are built up from these atomic formulae by the usual connectors $\wedge$ and $\neg$ and by existential quantification over elementary and over set variables. More precisely, if φ and ψ are formulae, then so are $\neg\varphi$, $\varphi \wedge \psi$, $\exists x_i \varphi$ and $\exists X_i \varphi$ where $i \in \mathbb{N}$. To define when a Σ-labeled graph (V, E, λ) satisfies a formula, let $f_e : V_e \to V$ and $f_s : V_s \to 2^V$ be mappings. Then

$(V,E,\lambda) \models_{f_e,f_s} E(x_i,x_j)$	iff	$(f_e(x_i), f_e(x_j)) \in E$,
$(V,E,\lambda) \models_{f_e,f_s} X_j(x_i)$	iff	$f_e(x_i) \in f_s(X_j)$,
$(V,E,\lambda) \models_{f_e,f_s} \lambda(x_i) = a$	iff	$\lambda \circ f_e(x_i) = a$,
$(V,E,\lambda) \models_{f_e,f_s} \neg\varphi$	iff	not $(V,E,\lambda) \models_{f_e,f_s} \varphi$, and
$(V,E,\lambda) \models_{f_e,f_s} \varphi \wedge \psi$	iff	$(V,E,\lambda) \models_{f_e,f_s} \varphi$ and $(V,E,\lambda) \models_{f_e,f_s} \psi$.

Furthermore, $(V, E, \lambda) \models_{f_e,f_s} \exists x_i \varphi$ if there exists a function $g_e : V_e \to V$ such that $(V, E, \lambda) \models_{g_e,f_s} \varphi$ and this function differs from f_e at most in the value of x_i. Similarly, $(V, E, \lambda) \models_{f_e,f_s} \exists X_j \varphi$ if there exists a function $g_s : V_s \to 2^V$ such that $(V, E, \lambda) \models_{f_e,g_s} \varphi$ and this function differs from f_s at most in the value of X_j.

Let (V, E, λ) be a Σ-labeled graph and let φ be a formula whose free variables are among $\{x_0, x_1, \ldots, x_k, X_0, X_1, \ldots, X_\ell\}$. Let furthermore $f_e, g_e : V_e \to V$ and $f_s, g_s : V_s \to 2^V$ be mappings such that $f_e(x_i) = g_e(x_i)$ for $0 \leq i \leq k$ and $f_s(X_i) = g_s(X_i)$ for $0 \leq i \leq \ell$. Then it is an easy exercise to show that $(V, E, \lambda) \models_{f_e,f_s} \varphi$ iff $(V, E, \lambda) \models_{g_e,g_s} \varphi$. For this reason, one usually writes

$$(V, E, \lambda) \models \varphi[f_e(x_0), f_e(x_1), \ldots, f_e(x_k), f_s(X_0), f_s(X_1), \ldots, f_s(X_\ell)]$$

for $(V, E, \lambda) \models_{f_e,f_s} \varphi$.

A formula without free variables is called *sentence*. Since the satisfaction of a sentence by a graph does not depend on the functions f_e and f_s, we will in this case simply say that the sentence holds in the graph. A formula is an *elementary formula* if it does not contain any set variable. To stress that some formula is not elementary, we will speak of *monadic formulas*, too.

Let (V, E, λ) be some Σ-labeled graph. The *elementary theory* $\mathrm{Th}(V, E, \lambda)$ of this graph is the set of all elementary sentences that hold in (V, E, λ). Similarly, the *monadic theory* $\mathrm{MTh}(V, E, \lambda)$ is the set of all monadic sentences valid in the graph. We also define the elementary and monadic theory of classes of Σ-labeled graphs $\mathbb{C}$ by

$$\begin{aligned} \mathrm{Th}(\mathbb{C}) &= \bigcap_{(V,E,\lambda)\in\mathbb{C}} \mathrm{Th}(V, E, \lambda), \text{ and} \\ \mathrm{MTh}(\mathbb{C}) &= \bigcap_{(V,E,\lambda)\in\mathbb{C}} \mathrm{MTh}(V, E, \lambda), \end{aligned}$$

i.e. the elementary (monadic, respectively) theory of a class of graphs is the set of all elementary (monadic, respectively) sentences that hold in all graphs of this class.

Let $\mathbb{C}_1 \subseteq \mathbb{C}_2$ be two classes of Σ-labeled graphs. The class $\mathbb{C}_1$ is *monadically axiomatizable relative to* $\mathbb{C}_2$ iff there exists a monadic sentence φ such that for any $(V, E, \lambda) \in \mathbb{C}_2$ we have $(V, E, \lambda) \in \mathbb{C}_1$ iff $(V, E, \lambda) \models \varphi$. If φ is even an elementary sentence, the class $\mathbb{C}_1$ is *elementarily axiomatizable*. Thus, the notion "axiomatizable" always refers to classes of graphs. Differently, the notion "definable" refers to relations inside some graph: Let $G = (V, E, \lambda)$ be a Σ-labeled graph, $n \in \mathbb{N}$ and φ be a monadic sentence whose free variables are among $\{x_0, x_1, \ldots, x_{n-1}\}$. Then

$$\varphi^G := \{(v_0, v_1, \ldots, v_{n-1}) \in V^n \mid G \models \varphi[v_0, v_1, \ldots, v_{n-1}]\}$$

is the *n-ary relation defined by φ*. An n-ary relation $R \subseteq V^n$ is *monadically definable inside G* if $R = \varphi^G$ for some monadic formula φ. *Elementarily definable relations* are defined similarly.

Later, we will also use logical formulae to reason on unlabeled graphs. It should be clear that this just requires that atomic formulas of the form $\lambda(x) = a$ do not occur in the formula in question. The notions satisfaction, sentence, elementary and monadic theory etc. then are the obvious restrictions of the notions we defined above. In the last chapters of both parts, we will concentrate on (labeled) partially ordered sets which are special (labeled) graphs. To make the formulas more intuitive, we will occasionally use subformulas of the form $x \leq y$ as a substitute for $E(x, y)$ and $x \in X$ for $X(x)$. Recall that in the definition of the satisfaction of a monadic formula, monadic variables range over arbitrary sets. Therefore, we considered functions $f_s : V_s \to 2^V$. If $(V, \leq)$ is a partially ordered set, one can restrict the monadic variables to range over chains or antichains, only. This is done by considering functions $f_s : V_s \to 2^V$ where $f_s(X)$ is an (anti-)chain for any $X \in V_s$. The resulting satisfaction relations are denoted by $\models_A$ if set variables range over antichains, and by $\models_C$ if the set variables range over chains, only. The *monadic chain theory* and the *monadic antichain theory* are defined canonically by

$$\begin{aligned}
\mathrm{MATh}(V, \leq) &= \{\varphi \text{ monadic sentence without } \lambda(x) = a \mid (V, \leq) \models_A \varphi\}\\
\mathrm{MCTh}(V, \leq) &= \{\varphi \text{ monadic sentence without } \lambda(x) = a \mid (V, \leq) \models_C \varphi\}\\
\mathrm{MATh}(\mathfrak{P}) &= \bigcap_{(V,E,\lambda)\in\mathfrak{P}} \mathrm{MATh}(V, E, \lambda), \text{ and}\\
\mathrm{MCTh}(\mathfrak{P}) &= \bigcap_{(V,E,\lambda)\in\mathfrak{P}} \mathrm{MCTh}(V, E, \lambda),
\end{aligned}$$

where $\mathfrak{P}$ is any set of posets.

As usual, we will use abbreviations like

$$\begin{aligned} \varphi \vee \psi &\quad \text{for} \quad \neg(\neg\varphi \wedge \neg\psi), \\ \varphi \rightarrow \psi &\quad \text{for} \quad \neg\varphi \vee \psi, \text{ and} \\ \forall x \varphi &\quad \text{for} \quad \neg\exists x \neg\varphi. \end{aligned}$$

Finally, for some properties that can obviously be expressed by a monadic formula, we will simply use their mathematical or English description as for instance "$\bigcup_{t \in T} X_t$ is everything" for "$\forall x \bigwedge_{t \in T} X_t(x)$" where T is a finite set or "X is a chain" for "$\forall x \forall y((X(x) \wedge X(y)) \rightarrow (E(x, y) \vee E(y, x)))$" .

1.4 Some notations

This very last part of the first chapter is devoted to some technical notions that will be used throughout this work. Most of them are standard in one or the other community but might be not so usual in another.

Let A, B be sets and $f : A \rightarrow B$. By $\operatorname{im} f$, we denote the image of f, i.e. the set $\{f(a) \mid a \in A\} \subseteq B$. The identity function $A \rightarrow A$ is denoted by id_A while the identity relation on A is $\Delta_A = \{(a, a) \mid a \in A\}$. For $A' \subseteq A$, a function $f : A' \rightarrow B$ is a partial function from A to B. The set A' is the domain $\mathrm{dom}(f)$ of the partial function f. By $\mathrm{part}(A, B)$, we denote the set of partial functions from A to B with nonempty domain. Already in the preceding section, I used the symbol 2^A for the powerset of A. By $\pi_1 : A \times B \rightarrow A$, we denote the projection to the first component of the direct product $A \times B$. Similarly, $\pi_2 : A \times B \rightarrow B$ is the projection to the second component. Finally, we write $[n] = \{1, 2, \ldots, n\}$ for the set of positive integers up to n while $\underline{n}$ denotes the set $\{0, 1, 2, \ldots, n - 1\}$.