Chapter 3

Decidability results

In the preceding chapter, we defined Σ-ACMs as a model of a computing device that can perform computation tasks concurrently. The behavior of a Σ-ACM is the accepted language, i.e. a set of Σ-dags. Hence a Σ-ACM describes a property of Σ-dags. Since the intersection as well as the union of two languages $L(\mathcal{A}_1)$ and $L(\mathcal{A}_2)$ can be accepted by a Σ-ACM, properties describable by Σ-ACMs can become quite complex. Then it is of interest whether the combined property is contradictory, or, equivalently, whether at least one Σ-dag satisfies it. Thus, one would like to know whether a Σ-ACM accepts at least one Σ-dag. In this chapter, we show that it is possible to gain this knowledge even automatically, i.e. we show that there exists an algorithm that on input of a Σ-ACM decides whether the Σ-ACM accepts at least one Σ-dag. In other words, the aim of this chapter is to show that the question "Does $\mathcal{A}$ accept some Σ-dag?" is decidable. More precisely, it is shown that the set

$$\{\mathcal{A} \mid \mathcal{A} \text{ is monotone and effective and } L(\mathcal{A}) = \emptyset\}$$

is recursive. I am grateful to Peter Habermehl who pointed me to the paper [FS98a, FS98b] that deals with well-structured transition systems. The proof of the mentioned decidability rests on this result.

3.1 Notational conventions and definitions

Let $\mathbb{N}^+ = \{1, 2, \dots\}$. Nevertheless, in this chapter an expression $\sup(M)$ for $M \subseteq \mathbb{N}^+$ will be understood in the structure $(\mathbb{N}, \leq)$. The useful effect of this convention is that $\sup(M) = 0$ for $M \subseteq \mathbb{N}^+$ if and only if M is empty.

Let A be a set. Then in this chapter, a *word* is a mapping $w : M \to A$ where M is a finite subset of $\mathbb{N}^+$. If $M = \{n_1, n_2, \dots, n_k\}$ with $n_1 < n_2 < \dots < n_k$, the finite sequence $w(n_1)w(n_2)\dots w(n_k)$ is a word in the usual sense. Two words $v : M \to A$ and $w : N \to A$ are *isomorphic* (and we will identify them) if there is an order isomorphism (with respect to the usual linear order of the natural

$$\begin{array}{ccc} s' & \longrightarrow & t' \\ \curlyvee\!| & & \curlyvee\!| \\ s & \longrightarrow & t \end{array}$$

Figure 3.1: Lifting of a transition in a WSTS

numbers) $\eta : M \to N$ with $v = w \circ \eta$. By $A^\star$ we denote the set of all words over A. Furthermore, for $w \in A^\star$ and $a \in A$ let wa denote the word $v : \operatorname{dom} w \cup \{n\} \to A$ with $n > \operatorname{dom} w$, $v \restriction \operatorname{dom} w = w$ and $v(n) = a$. By ε, we denote the empty word, i.e. the mapping $\varepsilon : \emptyset \to A$.

Recall that we identify isomorphic Σ-dags. Hence, we can impose additional requirements on the carrier set V as long as they can be satisfied in any isomorphism class. It turns out that in the considerations we are going to do in this section, it will be convenient to assume that for any Σ-dag (V, E, λ)

> $V \subseteq \mathbb{N}^+$ and that the partial order $E^\star$ is contained in the usual linear order on $\mathbb{N}^+$.

Note that on the set $H := \lambda^{-1}(a)$ we have two linear orders: $E^\star$ and the order $\leq$ of the natural numbers. Since $\leq$ extends $(V, E^\star)$, these two linear orders on H coincide. Hence, for a run r of some Σ-ACM on $t = (V, E, \lambda)$, the mapping $r \restriction \lambda^{-1}(a) : \lambda^{-1}(a) \to Q_a$ is a word over Q_a whose letters occur in the order given by $(V, E^\star)$.

3.2 Well-structured transition systems

A *transition system* is a set S endowed with a binary relation $\to \subseteq S^2$. For $t \in S$, we denote by $\operatorname{Pred}(t)$ the set of *predecessors* of t in the transition system S, i.e. the set of all $s \in S$ with $s \to t$. A *well-structured transition system* or *WSTS* is a triple $(S, \to, \preceq)$ where $(S, \to)$ is a transition system, $\preceq$ is a wqo on S and for any $s, s', t \in S$ with $s \to t$ and $s \preceq s'$ there exists $t' \in S$ with $s' \to t'$ and $t \preceq t'$. Thus, a WSTS is a well-quasi ordered transition system such that any transition $s \to t$ "lifts" to a larger state $s' \succeq s$ (cf. Figure 3.1). This definition differs slightly from the original one by Finkel & Schnoebelen [FS98b] in two aspects: First, they require only $s' \to^\star t'$ and they call a WSTS satisfying our axiom "WSTS with strong compatibility". Secondly, and more seriously, their transition systems are finitely branching. But it is easily checked that the results from [FS98b, Section 2 and 3] hold for infinitely branching transition systems, too. Since we use only these results (namely Theorem 3.6), it is not necessary to restrict well-structured transitions systems in our context to finitely branching ones. In [FS98b], several decidability results are shown for WSTSs. In particular, they showed

Theorem 3.2.1 ([FS98b, Theorem 3.6]) *Let $(S, \to, \preceq)$ be a WSTS such that $\preceq$ is decidable and a finite basis of* $\mathrm{Pred}(\uparrow s)$ *can be computed effectively for any $s \in S$. Then there is an algorithm that solves the following decision problem:*
input: *two states $s, t \in S$.*
output: *Does there exist a state $t' \in S$ with $s \to^\star t' \succeq t$, i.e. is t dominated by some state reachable from s?*

Since in their proof the algorithm that decides the existence of the state t' is uniformly constructed from the decision algorithm for $\preceq$ and the algorithm that computes a finite predecessor basis, one gets even more:

Theorem 3.2.2 *There exists an algorithm that solves the following decision problem:*
input: *1. an algorithm that decides $\preceq$,*
2. an algorithm computing a finite basis for $\mathrm{Pred}(\uparrow s)$ *for $s \in S$, and*
3. two states s and t from S
for some well-structured transition system $(S, \to, \preceq)$.
output: *Does there exists a state $t' \in S$ such that $s \to^\star t' \succeq t$?*

In this section, we will show that there is an algorithm that, given a Σ-ACM, decides whether this ACM accepts some Σ-dag. To obtain this result we use well-structured transition systems introduced above and in particular Theorem 3.2.2. Of course, the first idea might be to define a transition system as follows: The states are the runs of the Σ-ACM $\mathcal{A}$, i.e. we could define the state set Z to equal $\{(t, r) \mid t \in \mathbb{D}$ and r is a run of $\mathcal{A}$ on $t\}$. The transitions of the transition system should reflect the computation steps of the ACM $\mathcal{A}$, i.e. we could define $(t, r) \rightsquigarrow (t', r')$ iff there exists a maximal vertex x of t' such that $t = t' \setminus \{x\}$ and $r = r' \restriction t$. Then $(Z, \rightsquigarrow)$ is indeed a transition system that mimics the computations of the ACM $\mathcal{A}$. But to make it a well-structured transition system, we need a well-quasi order on Z that is compatible with $\rightsquigarrow$. Since the states of this transition system are labeled *graphs*, one could try the minor relation that is a wqo on *unlabeled* graphs. But (at least to the author) it is not clear whether this can be extended to labeled graphs (it is even unclear what the labeling of a minor should be).

Recall that the transition relation of the WSTS should reflect the atomic computation steps of the Σ-ACM $\mathcal{A}$. But the labeled graph (t, r) contains much information that is not necessary for this purpose. The only information we really need is

1. the state sequence of the a-component of the automaton $\mathcal{A}$, i.e. the Q_a-word $r \restriction \lambda^{-1}(a)$, and

2. which nodes of t can be read by an additional node x, i.e. for each $a, b \in \Sigma$ we need the information which a-labeled node has already been read by some b-labeled node.

For a Σ-ACM $\mathcal{A} = ((Q_a, \sqsubseteq_a)_{a\in\Sigma}, (\delta_{a,J})_{a\in\Sigma,J\subseteq\Sigma}, F)$, this idea is formalized as follows: Let $t = (V, E, \lambda)$ be a Σ-dag and let $r : V \to Q$ be a run of $\mathcal{A}$ on t. For $a \in \Sigma$, let $v_a := r \restriction \lambda^{-1}(a)$. As explained in Section 3.1, $\lambda^{-1}(a)$ is a subset of $\mathbb{N}^+$ where the order relation $E^\star$ coincides with the usual linear order $\leq$ on $\mathbb{N}$. Hence $v_a : \lambda^{-1}(a) \to Q_a$ is a word over Q_a. Now we define mappings $\mathrm{pos}_a^v : \Sigma \to V$ as follows: For $a, b \in \Sigma$, let $\mathrm{pos}_a^v(b)$ denote the last position in the word v_a that is read by some b-labeled vertex. Formally

$$\mathrm{pos}_a^v(b) := \sup\{x \in \lambda^{-1}(a) \mid \exists y \in \lambda^{-1}(b) : (x, y) \in E\}$$

where the supremum is taken in $\mathbb{N}$ such that we have $\mathrm{pos}_a^v(b) = 0$ iff the set is empty. Note that $\mathrm{pos}_a^v(b)$ is in general not the last position in v_a that is *dominated* by some b-labeled vertex in the partial order $(V, E^\star, \lambda)$. The tuple $(v_a, \mathrm{pos}_a^v)_{a\in\Sigma}$ is called the *state associated with the run* r, denoted $\mathrm{state}(r) := (v_a, \mathrm{pos}_a^v)_{a\in\Sigma}$.

Example 2.1.3 (continued) Let $t = (V, E, \lambda)$ be the Σ-dag and let r denote the run of $\mathcal{A}$ depicted in Figure 2.3 (page 15). Then we have the following:

$$\begin{aligned}
v_a &= (1,0,0)(2,0,0)(3,0,0)\\
v_b &= (1,1,0)(2,2,0)\\
v_c &= (1,1,1)(1,1,2)\\
\mathrm{pos}_a^v &= \{(a,2),(b,2),(c,0)\}\\
\mathrm{pos}_b^v &= \{(a,0),(b,1),(c,1)\}\\
\mathrm{pos}_c^v &= \{(a,0),(b,0),(c,1)\}
\end{aligned}$$

This situation is visualized in Figure 3.2. There, the words v_a, v_b and v_c are drawn vertically. On the left of a node, the associated state of $\mathcal{A}$ can be found. The letter b at the right of the second a-node indicates that this node equals $\mathrm{pos}_a^v(b)$. Finally, $\mathrm{pos}_a^v(c) = 0$ is indicated by the fact that c does not appear at the right of the word v_a.

As explained above, we want the set of states S to contain $\mathrm{state}(r)$. Thus, $S \subseteq \prod_{a\in\Sigma}(Q_a^\star \times \mathbb{N}^\Sigma)$. Now we define the state set S completely:

$$S := \left\{ (v_a, \mathrm{pos}_a^v)_{a\in\Sigma} \in \prod_{a\in\Sigma}(Q_a^\star \times \mathbb{N}^\Sigma) \mid \mathrm{im}(\mathrm{pos}_a^v) \subseteq \mathrm{dom}\, v_a \cup \{0\} \text{ for } a \in \Sigma \right\}.$$

Note that $0 \notin \mathbb{N}^+$ and therefore in general $\mathrm{im}\,\mathrm{pos}_a^v \not\subseteq \mathrm{dom}\, v_a$.

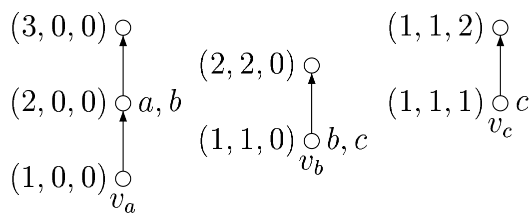

Figure 3.2: The state state(r) of the run from Figure 2.3

The state $(w_a, \text{pos}_a^w)_{a\in\Sigma}$ is a *successor* of the state $(v_a, \text{pos}_a^v)_{a\in\Sigma}$, denoted $(v_a, \text{pos}_a^v)_{a\in\Sigma} \to (w_a, \text{pos}_a^w)_{a\in\Sigma}$, iff there exist $a \in \Sigma$, $\emptyset \neq J \subseteq \Sigma$, $p_b \in Q_b$ for $b \in J$ and $q \in Q_a$ such that

(i) $q \in \delta_{a,J}((p_b)_{b\in J})$,

(ii) $w_c = \begin{cases} v_c q & \text{for } c = a \\ v_c & \text{otherwise,} \end{cases}$

(iii) $\text{pos}_c^w(b) = \text{pos}_c^v(b)$ for all $b, c \in \Sigma$ satisfying $c \notin J$ or $a \neq b$, and

(iv) $\text{pos}_c^v(a) < \text{pos}_c^w(a) \in \text{dom}\, v_c$ such that $v_c \circ \text{pos}_c^w(a) = p_c$ for $c \in J$.

In this chapter, we will refer to these conditions just as (i),(ii) etc.

The following example indicates that state$(r) \to$ state(r') whenever $(t, r) \rightsquigarrow (t', r')$, i.e. that the transition system $(S, \to)$ really reflects the computations of the ACM $\mathcal{A}$. Even more, we will show that (under some additional assumptions on $\mathcal{A}$) the states of the form state(r) for some run r are precisely those states that are *"reachable"* in the transition system $(S, \to)$ (cf. Lemma 3.3.3). This will enable us to prove the desired decidability result.

Example 2.1.3 (continued) Let t' denote the extension of the Σ-dag t from Figure 2.3 by an a-labeled node as indicated in Figure 3.3 (first picture). Furthermore, this picture shows an extension r' of the run r, too. The second picture depicts the state state(r'). The reader might check that state(r') is a successor state of state(r).

First, we will show that the result of Finkel & Schnoebelen can indeed be applied, i.e. that we can extend the transition system $(S, \to)$ to a well-structured transition system.

So we have to extend the wqo $\sqsubseteq_a$ on Q_a to words over Q_a: To do this, recall that we consider words as mappings from a finite linear order into the well-quasi

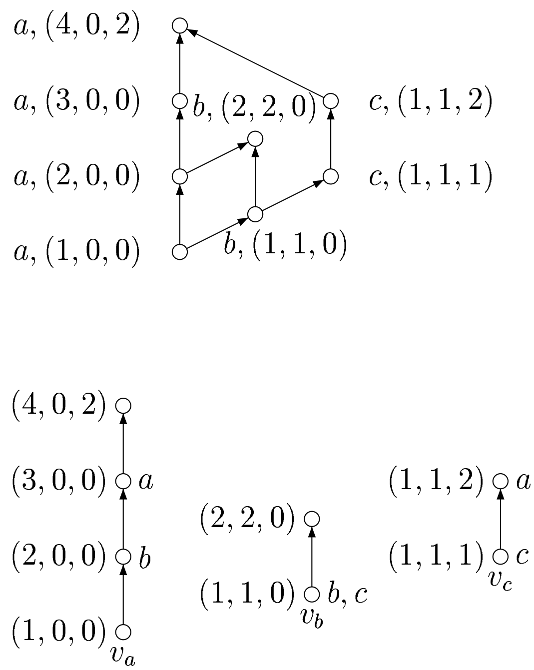

Figure 3.3: A successor state of state(r) from Figure 3.2

ordered set Q_a. Therefore, an *embedding* $\eta : v \hookrightarrow w$ is defined to be an order embedding of $\operatorname{dom} v \cup \{0\}$ into $\operatorname{dom} w \cup \{0\}$ such that

$$\eta(0) = 0,\ \eta(\sup \operatorname{dom} v) = \sup \operatorname{dom} w,\ \text{and } v(i) \sqsubseteq_a w \circ \eta(i) \text{ for } i \in \operatorname{dom} v.$$

Thus, there is an embedding $\eta : v \hookrightarrow w$ iff one obtains v from w by first deleting some letters (but not the last) and then decreasing the remaining ones with respect to $\sqsubseteq_a$. If $\sqsubseteq_a$ is trivial (i.e. the identity relation Δ_{Q_a}), there exists such an embedding iff v is a subword of w and the last letters of v and w coincide. Now a quasi-order $\preceq$ on the states of the transition system $(S, \rightarrow)$ is defined by $(v_a, \operatorname{pos}_a^v)_{a\in\Sigma} \preceq (w_a, \operatorname{pos}_a^w)_{a\in\Sigma}$ iff

> there exist embeddings $\eta_a : v_a \hookrightarrow w_a$ such that $\eta_a \circ \operatorname{pos}_a^v = \operatorname{pos}_a^w$ for any $a \in \Sigma$.

As explained above, the existence of the embeddings η_a ensures that v_a is dominated by some subword (including the last letter) of w_a letter by letter. The requirement $\eta_a \circ \operatorname{pos}_a^v = \operatorname{pos}_a^w$ ensures that the pointer $\operatorname{pos}_a^w(b)$ (if not 0) points to some position in this *subword* and that this position corresponds (via η_a) to the position in v_a to which $\operatorname{pos}_a^v(b)$ points. It is obvious that $\preceq$ is reflexive and transitive, i.e. $\preceq$ is a quasiorder. If $\sqsubseteq_a$ is a partial order for any $a \in \Sigma$, the relation $\preceq$ is even a partial order since $(v_a, \operatorname{pos}_a^v)_{a\in\Sigma} \preceq (w_a, \operatorname{pos}_a^w)_{a\in\Sigma}$ implies $|v_a| \leq |w_a|$ for any $a \in \Sigma$.

Lemma 3.2.3 *Let $\mathcal{A}$ be a Σ-ACM. Then $(S, \preceq)$ is a well quasi ordering.*

Proof. Let $w \in Q^\star$ and $\operatorname{pos}^w : \Sigma \to \operatorname{dom} w$. We construct a word w' over the set $Q \times 2^\Sigma$ by $\operatorname{dom} w' := \operatorname{dom} w$ and $w(i) := (w(i), (\operatorname{pos}^w)^{-1}(i))$. Now let $v \in Q^\star$ and $\operatorname{pos}^v : \Sigma \to \operatorname{dom} v$ and construct $v' \in (Q \times 2^\Sigma)^\star$ similarly. Then there is an embedding $\eta : v \hookrightarrow w$ with $\eta \circ \operatorname{pos}^v = \operatorname{pos}^w$ iff there exists an embedding $\eta' : v' \hookrightarrow w'$. Thus, $(v_a, \operatorname{pos}_a^v)_{a\in\Sigma} \preceq (w_a, \operatorname{pos}_a^w)_{a\in\Sigma}$ iff v'_a can be embedded into w'_a for any $a \in \Sigma$.

By Higman's Theorem [Hig52], words over a well-quasi ordered set $(Q, \sqsubseteq)$ form a wqo with respect to the embeddability. Since the direct product of finitely many wqos is a wqo, the lemma follows. □

For any Σ-ACM, the structure $(S, \rightarrow)$ is a transition system whose set of states is equipped with the well-quasi order $\preceq$. if the Σ-ACM is monotone, the triple $(S, \rightarrow, \preceq)$ is a well-structured transition system.

Theorem 3.2.4 *Let $\mathcal{A}$ be a monotone Σ-ACM. Then $\mathcal{S}(\mathcal{A}) = (S, \rightarrow, \preceq)$ is a well-structured transition system.*

Proof. Let $(v_c, \mathrm{pos}_c^v)_{c\in\Sigma}$, $(w_c, \mathrm{pos}_c^w)_{c\in\Sigma}$ and $(v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma}$ be states from S such that

$$\begin{array}{ccc} (v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma} & & \\ \curlyvee\!| & & \\ (v_c, \mathrm{pos}_c^v)_{c\in\Sigma} & \to & (w_c, \mathrm{pos}_c^w)_{c\in\Sigma}. \end{array}$$

Let $\eta_c : v_c \hookrightarrow v'_c$ denote embeddings that witness $(v_c, \mathrm{pos}_c^v)_{c\in\Sigma} \preceq (v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma}$. Since $(v_c, \mathrm{pos}_c^v)_{c\in\Sigma} \to (w_c, \mathrm{pos}_c^w)_{c\in\Sigma}$, there exist $a \in \Sigma$, $\emptyset \neq J \subseteq \Sigma$, $p_b \in Q_b$ for $b \in J$, and $q \in Q_a$ satisfying (i)-(iv).

In particular (by (i)) $q \in \delta_{a,J}((p_b)_{b\in J})$. By (iv), we get in addition $p_c = v_c(\mathrm{pos}_c^v(a)) \sqsubseteq_c v'_c \circ \eta_c(\mathrm{pos}_c^{v'}(a)) =: p'_c$. Hence, by the monotonicity of the Σ-ACM $\mathcal{A}$, there exists $q' \in \delta_{a,J}((p'_b)_{b\in J})$ such that $q \sqsubseteq_a q'$.

Let $w'_a := v'_a q'$ and $w'_c := v'_c$ for $c \neq a$. Extend η_a to η'_a by $\eta'_a := \eta_a \cup \{(\sup\mathrm{dom}\, w_a, \sup\mathrm{dom}\, w'_a)\}$ and $\eta'_c := \eta_c$ for $c \neq a$ and define $\mathrm{pos}_c^{w'} := \eta'_c \circ \mathrm{pos}_c^w$ for $c \in \Sigma$. Then $(w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma} \in S$ and $(w_c, \mathrm{pos}_c^w)_{c\in\Sigma} \preceq (w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma}$ witnessed by the embeddings η'_c. It remains to show that $(w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma}$ is a successor of $(v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma}$, i.e. we have to prove that (i)-(iv) hold for $(v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma}$, $(w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma}$, a, J and p'_b for $b \in J$ and q': Property (ii) follows from the definition of w'_c. Now let $b, c \in \Sigma$ with $c \notin J$ or $a \neq b$. Then $\mathrm{pos}_c^{w'}(b) = \eta'_c \circ \mathrm{pos}_c^w(b) = \eta'_c \circ \mathrm{pos}_c^v(b)$ since (iii) holds for the undashed elements. Since $\mathrm{pos}_c^v(b) \in \mathrm{dom}\, v_c$, we have $\eta'_c \circ \mathrm{pos}_c^v(b) = \eta_c \circ \mathrm{pos}_c^v(b) = \mathrm{pos}_c^{v'}(b)$, i.e. we showed (iii). To verify (iv), let $c \in J$. Then $\mathrm{pos}_c^{v'}(a) = \eta_c \circ \mathrm{pos}_c^v(a) = \eta'_c \circ \mathrm{pos}_c^v(a) < \eta'_c \circ \mathrm{pos}_c^w(a)$ since (iv) holds for the undashed elements and η'_c is an order embedding. Since $\eta'_c \circ \mathrm{pos}_c^w(a) = \mathrm{pos}_c^{w'}(a)$, we get $\mathrm{pos}_c^{v'}(a) < \mathrm{pos}_c^{w'}(a) \in \mathrm{im}\, \eta_c$. Since $\mathrm{pos}_c^w(a) \neq 0$ and η'_c is injective, we obtain $\mathrm{pos}_c^{w'}(a) = \eta'_c \circ \mathrm{pos}_c^w(a) \neq 0$, i.e. $\mathrm{pos}_c^{w'}(a) \in \mathrm{im}\, \eta_c \setminus \{0\} \subseteq \mathrm{dom}\, v'_c$. Finally, $v'_c \circ \mathrm{pos}_c^{v'}(a) = p'_c$ holds by the choice of p'_c. □

To apply Theorem 3.2.2 to the WSTS $(S, \to, \preceq)$, our next aim is to show that in $(S, \to, \preceq)$ a finite predecessor basis, i.e. a finite basis of $\mathrm{Pred}(\uparrow(w_c, \mathrm{pos}_c^w)_{c\in\Sigma})$, can be computed for any state $(w_c, \mathrm{pos}_c^w)_{c\in\Sigma}$. Note that $\uparrow(w_c, \mathrm{pos}_c^w)_{c\in\Sigma}$ in this expression is meant with respect to the wqo $\preceq$. Before we can prove this (cf. Lemma 3.2.8), we consider the quasiorder $\sqsubseteq$ on S: For $v, w \in Q_a^\star$, let $v \sqsubseteq'_a w$ iff $|v| = |w|$ and there exists an embedding of v into w. Note that whenever $v \sqsubseteq'_a w$ we can obtain w from v by simply enlarging the letters of v independently from each other. Since comparable words (with respect to $\sqsubseteq'_a$) have the same length, $\sqsubseteq'_a$ is only a quasiorder, but not a wqo. Similarly to $\sqsubseteq'_a$, we define $(v_a, \mathrm{pos}_a^v)_{a\in\Sigma} \sqsubseteq (w_a, \mathrm{pos}_a^w)_{a\in\Sigma}$ iff $|v_a| = |w_a|$ for $a \in \Sigma$ and $(v_a, \mathrm{pos}_a^v)_{a\in\Sigma} \preceq (w_a, \mathrm{pos}_a^w)_{a\in\Sigma}$.

We call a Σ-ACM *effective* if there is an algorithm that given $a \in \Sigma$, $J \subseteq \Sigma$, $p_b \in Q_b$ for $b \in J$ and $q \in Q_a$ computes a finite basis of

$$\left\{ ((p'_b)_{b\in J}, q') \in \prod_{b\in J} Q_b \times Q_a \;\middle|\; q' \in \delta_{a,J}((p'_b)_{b\in J}), q \sqsubseteq_a q' \text{ and } p_b \sqsubseteq_b p'_b \text{ for } b \in J \right\}$$

with respect to the direct product $\left(\prod_{b\in J} \sqsubseteq_b\right) \times \sqsubseteq_a$. We call such an algorithm a *basis algorithm of* $\mathcal{A}$. Intuitively, an ACM is effective if a finite basis of all transitions above a given tuple of states can be computed. Note that this tuple is not necessarily a transition. On the other hand, we do not require that the set of all transitions, i.e. the set $\{(q, (p_b)_{b\in J}) \mid q \in \delta_{a,J}((p_b)_{b\in J})\}$ is a recursive subset of $Q_a \times \prod_{b\in J} Q_b$, and this might not be the case as the following example shows. Furthermore note that any asynchronous cellular automaton is effective since (as a finite object) it can be given explicitly.

Example 3.2.5 Let $\Sigma = \{a\}$ and $Q_a = \mathbb{N}$. On this set, we consider the complete relation $\mathbb{N} \times \mathbb{N}$ as wqo $\sqsubseteq_a$. Furthermore, let M be some non recursive subset of $\mathbb{N}$ and define, for $n \in \mathbb{N}$:

$$\delta_{a,\{a\}}(n) = \begin{cases} \{n, n+1\} & \text{if } n \in M \\ \{n\} & \text{if } n \notin M. \end{cases}$$

Furthermore, let $\delta_{a,\emptyset} = \{1\}$. Now let $t = (V, E, \lambda)$ be a Σ-dag (i.e. t is the Hasse-diagram of a finite linear order) and let $r : V \to \mathbb{N}$ be some mapping. Then r is a run of the Σ-ACA $\mathcal{A} = (Q_a, (\delta_{a,J})_{J\subseteq\{a\}}, F)$ iff $r(x) \leq r(x+1) \leq r(x) + 1$ for any $x \in V$ and $\{x \in V \mid r(x) \neq r(x+1)\} \subseteq M$. Since this latter inclusion is not decidable, one cannot decide whether r is a run. On the other hand, $\mathcal{A}$ is effective since $\{(1,1)\}$ is a finite basis of any nonempty subset of $Q_a \times Q_a$.

The preceding example suggests the question whether $L(\mathcal{A})$ is recursive for any monotone and effective Σ-ACM $\mathcal{A}$. Later (Corollary 3.3.6), we will show that this is indeed the case. Anyway, for an effective ACM, we can show:

Lemma 3.2.6 *There is an algorithm that, on input of an alphabet Σ, a basis algorithm for an effective Σ-ACM $\mathcal{A}$ and a state $(w_c, \mathrm{pos}_c^w)_{c\in\Sigma} \in S$, outputs a finite basis with respect to $\sqsubseteq$ of the set of all states $(v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma} \in S$ satisfying*

$$\begin{array}{cccc} \exists (w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma} : & (v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma} & \longrightarrow & (w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma} \\ & & & \sqcup\!\!\mid \\ & & & (w_c, \mathrm{pos}_c^{w})_{c\in\Sigma} \end{array}$$

Proof. In this proof, we assume that $\mathrm{dom}\, w = \{1, 2, \ldots, \sup \mathrm{dom}\, w\}$ for any word $w \neq \varepsilon$.

First, we describe the algorithm:

For any $a \in \Sigma$ and any $\emptyset \neq J \subseteq \Sigma$ that satisfy
(a) $w_a \neq \varepsilon$ and $\sup\operatorname{dom} w_a \notin \operatorname{im}\operatorname{pos}_a^w$ and
(b) $\operatorname{pos}_b^w(a) \neq 0$ for $b \in J$
compute a finite basis $B(a, J)$ of the set of all tuples $((p'_b)_{b\in J}, q') \in \prod_{b\in J} Q_b \times Q_a$ satisfying
(c) $q' \in \delta_{a,J}((p'_b)_{b\in J})$, $w_a(\sup\operatorname{dom} w_a) \sqsubseteq_a q'$ and $w_b \circ \operatorname{pos}_b^w(a) \sqsubseteq_b p'_b$ for $b \in J$.
Such a finite basis can be computed by an application of the basis algorithm with $q = w_a(\sup\operatorname{dom} w_a)$ and $p_b = w_b(\operatorname{pos}_b^w(a))$ for $b \in J$.
For any $((p'_b)_{b\in J}, q') \in B(a, J)$, let $(w'_c, \operatorname{pos}_c^{w'})_{c\in\Sigma} \in S$ denote the uniquely determined state that satisfies
(d) $\operatorname{dom} w'_c = \operatorname{dom} w_c$ and $\operatorname{pos}_c^{w'} = \operatorname{pos}_c^w$ for $c \in \Sigma$ and
(e) $w'_c(i) = \begin{cases} p'_c & \text{if } c \in J, i = \operatorname{pos}_c^{w'}(a) \\ q' & \text{if } c = a, i = \sup\operatorname{dom} w'_a \\ w_c(i) & \text{otherwise.} \end{cases}$
For $c \in \Sigma$, let v'_c denote the word over Q_c uniquely determined by
(f) $\operatorname{dom} v'_c = \begin{cases} \operatorname{dom} w'_a \setminus \{\sup\operatorname{dom} w'_a\} & \text{for } c = a \\ \operatorname{dom} w'_c & \text{otherwise, and} \end{cases}$
(g) $v'_c = w'_c \restriction \operatorname{dom} v'_c$.
Finally, output the finite set of states $(v'_c, \operatorname{pos}_c^{v'})_{c\in\Sigma}$ that satisfy
(h) $\operatorname{pos}_c^{v'}(b) = \operatorname{pos}_c^{w'}(b)$ for $b, c \in \Sigma$ with $c \notin J$ or $a \neq b$ and
(j) $\operatorname{pos}_c^{v'}(a) < \operatorname{pos}_c^{w'}(a)$ for $c \in J$.

First we show that for any $(v'_c, \operatorname{pos}_c^{v'})_{c\in\Sigma}$ that is output by the algorithm above we have $(v'_c, \operatorname{pos}_c^{v'})_{c\in\Sigma} \to (w'_c, \operatorname{pos}_c^{w'})_{c\in\Sigma} \sqsupseteq (w_c, \operatorname{pos}_c^w)_{c\in\Sigma}$:

Since $(v'_c, \operatorname{pos}_c^{v'})_{c\in\Sigma}$ is output, there exist $a \in \Sigma$, $\emptyset \neq J \subseteq \Sigma$, $p'_b \in Q_b$ for $b \in J$ with $((p'_b)_{b\in J}, q') \in B(a, J)$ and $q' \in Q_a$ such that (a)-(j) hold. For $c \in \Sigma$, the identity function $\eta_c : \operatorname{dom} w_c \cup \{0\} \to \operatorname{dom} w'_c \cup \{0\}$ satisfies $\eta_c \circ \operatorname{pos}_c^w = \operatorname{pos}_c^{w'}$ by (d). By (c) and (e), we obtain $w_c(i) \sqsubseteq_c w'_c \circ \eta_c(i)$ for $i \in \operatorname{dom} w_c$. Hence $(w_c, \operatorname{pos}_c^w)_{c\in\Sigma} \preceq (w'_c, \operatorname{pos}_c^{w'})_{c\in\Sigma}$. Since in addition $|w_c| = |w'_c|$, we get $(w_c, \operatorname{pos}_c^w)_{c\in\Sigma} \sqsubseteq (w'_c, \operatorname{pos}_c^{w'})_{c\in\Sigma}$.

It remains to show that (i)-(iv) (page 23) hold for the states $(v'_c, \operatorname{pos}_c^{v'})_{c\in\Sigma}$ and $(w'_c, \operatorname{pos}_c^{w'})_{c\in\Sigma}$ and for a, J, p'_b for $b \in J$ and q':
(i): This is immediate since (c) holds.
(ii): If $c \neq a$, (f) and (g) imply $v'_c = w'_c$. Furthermore, these two statements also ensure $w'_a = v'_a\, w'_a(\sup\operatorname{dom} w'_a) = v'_a\, q'$ by (e).
(iii): This is immediate by (h).
(iv): Let $c \in J$. Then, by (j), $\operatorname{pos}_c^{v'}(a) < \operatorname{pos}_c^{w'}(a)$. Hence $\operatorname{pos}_c^{w'}(a) \neq 0$ and therefore $\operatorname{pos}_c^{w'}(a) \in \operatorname{dom} w'_c$. For $c \neq a$, this implies $\operatorname{pos}_c^{w'} \in \operatorname{dom} v'_c$ since, by (f), $\operatorname{dom} v'_c = \operatorname{dom} w'_c$. To deal with the case $a = c$, recall that $\operatorname{pos}_a^w(a) \neq \sup\operatorname{dom} w_a$ by (a). Hence, from (d), we can infer $\operatorname{pos}_y^{w'}(a) \neq \sup\operatorname{dom} w'_a$ and therefore $\operatorname{pos}_a^{w'}(a) \in \operatorname{dom} w'_a \setminus \{\sup\operatorname{dom} w'_a\} = \operatorname{dom} v'_a$ by (f). Thus, we showed $\operatorname{pos}_c^{v'}(a) < \operatorname{pos}_c^{w'}(a) \in \operatorname{dom} v'_c$ for $c \in J$. Again, let $c \in J$. Then $v'_c(\operatorname{pos}_c^{w'}(a)) = w'_c(\operatorname{pos}_c^{w'}(a)) = p'_c$ by (e).

It remains to show that a state $(v''_c, \mathrm{pos}_c^{v''})_{c\in\Sigma} \in S$ dominates some output of our algorithm whenever there exists a state $(w''_c, \mathrm{pos}_c^{w''})_{c\in\Sigma} \in S$ such that:

$$\begin{array}{ccc} (v''_c, \mathrm{pos}_c^{v''})_{c\in\Sigma} & \longrightarrow & (w''_c, \mathrm{pos}_c^{w''})_{c\in\Sigma} \\ & & \sqcup\!\!\mid \\ & & (w_c, \mathrm{pos}_c^{w})_{c\in\Sigma} \end{array}$$

Since $(v''_c, \mathrm{pos}_c^{v''})_{c\in\Sigma} \to (w''_c, \mathrm{pos}_c^{w''})_{c\in\Sigma}$, there exist $a \in \Sigma$, $\emptyset \neq J \subseteq \Sigma$, $p''_b \in Q_b$ for $b \in J$ and $q'' \in Q_a$ satisfying (i)-(iv). We show that a and J satisfy (a) and (b):
(a): Since $(w''_c, \mathrm{pos}_c^{w''})_{c\in\Sigma} \sqsupseteq (w_c, \mathrm{pos}_c^{w})_{c\in\Sigma}$, it holds $|w_a| = |w''_a| = |v''_a q''| > 0$ by (ii) and therefore $w_a \neq \varepsilon$. Furthermore, $\mathrm{im}\,\mathrm{pos}_a^{w''} \subseteq \mathrm{dom}\, v''_a \cup \{0\}$ by (iii) and (iv). But $\mathrm{dom}\, v''_a = \mathrm{dom}\, w''_a \setminus \{\sup \mathrm{dom}\, w''_a\}$ by (ii) and therefore $\sup \mathrm{dom}\, w_a \notin \mathrm{im}\,\mathrm{pos}_a^{w}$.
(b): Let $c \in J$. Then $\mathrm{pos}_c^{w''}(a) \in \mathrm{dom}\, v''_c \not\ni 0$ by (iv). Hence $\mathrm{pos}_c^{w''}(a) \neq 0$ which does not belong to $\mathrm{dom}\, v''_c$.

Furthermore note that $w_b(\mathrm{pos}_b^{w}(a)) \sqsubseteq_b w''_b(\mathrm{pos}_b^{w''}(a)) = v''_b(\mathrm{pos}_b^{w''}(a)) = p''_b$ by (iv) for any $b \in J$. Similarly, $w_a(\sup \mathrm{dom}\, w_a) \sqsubseteq_a w''_a(\sup \mathrm{dom}\, w''_a) = q''$ by (iv) and (by (i)) $q'' \in \delta_{a,J}((p''_b)_{b\in J})$. Since $B(a, J)$ is a basis, there is $((p'_b)_{b\in\Sigma}, q') \in B(a, J)$ such that $p'_b \sqsubseteq_b q''_b$ for $b \in J$, $q' \sqsubseteq_a q''$ and (c) holds. Now construct $(w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma}$ and v'_c for $c \in \Sigma$ according to (d)-(g) and set $\mathrm{pos}_c^{v'} = \mathrm{pos}_c^{v''}$. To show (h), let $b, c \in \Sigma$ with $c \notin J$ or $a \neq b$. Then, by (iii), $\mathrm{pos}_c^{v''}(b) = \mathrm{pos}_c^{w''}(b)$. Since $(w_d, \mathrm{pos}_d^{w})_{d\in\Sigma} \sqsubseteq (w''_d, \mathrm{pos}_d^{w''})_{d\in\Sigma}$ is witnessed by the identity functions, we get $\mathrm{pos}_c^{v''}(b) = \mathrm{pos}_c^{w''}(b) = \mathrm{pos}_c^{w}(b) = \mathrm{pos}_c^{w'}(b)$ where the last equality holds by (d). Thus, (h) holds. To show (j), let $c \in J$. Then, by (iv), $\mathrm{pos}_c^{v''}(a) < \mathrm{pos}_c^{w''}(a)$ and we can continue as above by $\mathrm{pos}_c^{v''}(a) < \mathrm{pos}_c^{w''}(a) = \mathrm{pos}_c^{w}(a) = \mathrm{pos}_c^{w'}(a)$ thereby proving (j). Hence $h := (v'_c, \mathrm{pos}_c^{v''})_{c\in\Sigma}$ is a state from S that is output by our algorithm. It remains to check $h \sqsubseteq (v''_c, \mathrm{pos}_c^{v''})_{c\in\Sigma}$ which is left to the interested reader. $\square$

Lemma 3.2.7 *Let* $(x_c, \mathrm{pos}_c^x)_{c\in\Sigma}, (w_c'', \mathrm{pos}_c^{w''})_{c\in\Sigma}$ *and* $(v_c'', \mathrm{pos}_c^{v''})_{c\in\Sigma}$ *be states from* S *with*

$$\begin{array}{ccc} (v_c'', \mathrm{pos}_c^{v''})_{c\in\Sigma} & \to & (w_c'', \mathrm{pos}_c^{w''})_{c\in\Sigma} \\ & & \curlyvee| \\ & & (x_c, \mathrm{pos}_c^x)_{c\in\Sigma} \end{array}$$

Then there exist states $(w_c, \mathrm{pos}_c^w)_{c\in\Sigma}$, $(v_c', \mathrm{pos}_c^{v'})_{c\in\Sigma}$ *and* $(w_c', \mathrm{pos}_c^{w'})_{c\in\Sigma}$ *such that*

1. $|w_c| - |x_c| \leq 2|\Sigma| + 1$ *for* $c \in \Sigma$ *and*
2. $$\begin{array}{ccc} (v_c'', \mathrm{pos}_c^{v''})_{c\in\Sigma} & \to & (w_c'', \mathrm{pos}_c^{w''})_{c\in\Sigma} \\ \curlyvee| & & \curlyvee| \\ (v_c', \mathrm{pos}_c^{v'})_{c\in\Sigma} & \to & (w_c', \mathrm{pos}_c^{w'})_{c\in\Sigma} \\ & & \sqcup| \\ & & (w_c, \mathrm{pos}_c^w)_{c\in\Sigma} \\ & & \curlyvee| \\ & & (x_c, \mathrm{pos}_c^x)_{c\in\Sigma} \end{array}$$

Proof. Since $(v_c'', \mathrm{pos}_c^{v''})_{c\in\Sigma} \to (w_c'', \mathrm{pos}_c^{w''})_{c\in\Sigma}$, there are $a \in \Sigma$, $\emptyset \neq J \subseteq \Sigma$, $p_b \in Q_b$ for $b \in J$ and $q \in Q_a$ such that (i)-(iv) (page 23) hold. Let $\eta_c : x_c \hookrightarrow w_c''$ be embeddings that witness $(x_c, \mathrm{pos}_c^x)_{c\in\Sigma} \preceq (w_c'', \mathrm{pos}_c^{w''})_{c\in\Sigma}$. We may assume that η_c is just the identity function, i.e. $\mathrm{dom}\, x_c \subseteq \mathrm{dom}\, w_c''$, $x_c(i) \sqsubseteq_c w_c''(i)$ for $i \in \mathrm{dom}\, x_c$, $\sup \mathrm{dom}\, x_c = \sup \mathrm{dom}\, w_c''$, and $\mathrm{pos}_c^x = \mathrm{pos}_c^{w''}$ for $c \in \Sigma$.

First, we define $(w_c', \mathrm{pos}_c^{w'})_{c\in\Sigma}$: For $c \in \Sigma$, let

$$\mathrm{dom}\, w_c' := (\mathrm{dom}\, x_c \cup \mathrm{im}\, \mathrm{pos}_c^{v''} \cup \mathrm{im}\, \mathrm{pos}_c^{w''} \cup \{\sup \mathrm{dom}\, v_c''\}) \setminus \{0\}$$

and $w_c' = w_c'' \restriction \mathrm{dom}\, w_c'$. Furthermore, let $\mathrm{pos}_c^{w'} := \mathrm{pos}_c^{w''} = \mathrm{pos}_c^x$. Then $\mathrm{im}\, \mathrm{pos}_c^{w'} \subseteq \mathrm{dom}\, x_c \cup \{0\} \subseteq \mathrm{dom}\, w_c' \cup \{0\}$ ensures $(w_c', \mathrm{pos}_c^{w'})_{c\in\Sigma} \in S$.

We show $(w_c', \mathrm{pos}_c^{w'})_{c\in\Sigma} \preceq (w_c'', \mathrm{pos}_c^{w''})_{c\in\Sigma}$: Note that $\mathrm{dom}\, x_c \subseteq \mathrm{dom}\, w_c''$. Furthermore, $\mathrm{im}\, \mathrm{pos}_c^{v''} \subseteq \mathrm{dom}\, v_c'' \cup \{0\}$ and $(v_c'', \mathrm{pos}_c^{v''})_{c\in\Sigma} \to (w_c'', \mathrm{pos}_c^{w''})_{c\in\Sigma}$ imply $\mathrm{im}\, \mathrm{pos}_c^{v''} \setminus \{0\} \subseteq \mathrm{dom}\, w_c''$. Since $\mathrm{im}\, \mathrm{pos}_c^{w''} \setminus \{0\} \subseteq \mathrm{dom}\, w_c''$ and $\sup \mathrm{dom}\, v_c'' \in \mathrm{dom}\, w_c'' \cup \{0\}$, we therefore get $\mathrm{dom}\, w_c' \subseteq \mathrm{dom}\, w_c''$. Thus, the identity function $\eta_c' := \mathrm{id}_{\mathrm{dom}\, w_c' \cup \{0\}} : \mathrm{dom}\, w_c' \cup \{0\} \to \mathrm{dom}\, w_c'' \cup \{0\}$ is an order embedding that satisfies $w_c'(i) = w_c'' \circ \eta_c'(i)$ for $i \in \mathrm{dom}\, w_c'$. Since $(v_c'', \mathrm{pos}_c^{v''})_{c\in\Sigma} \to (w_c'', \mathrm{pos}_c^{w''})_{c\in\Sigma}$, we have $\mathrm{dom}\, v_c'' \subseteq \mathrm{dom}\, w_c''$ and therefore $\sup \mathrm{dom}\, v_c'' \leq \mathrm{dom}\, \mathrm{dom}\, w_c'' = \sup \mathrm{dom}\, x_c$. Hence $\sup \mathrm{dom}\, w_c' = \sup \mathrm{dom}\, x_c = \sup \mathrm{dom}\, w_c''$. Thus, $\eta' : w_c' \hookrightarrow w_c''$ is an embedding. Since $\mathrm{pos}_c^{w'} = \mathrm{pos}_c^{w''}$, we in addition get $\mathrm{pos}_c^{w''} = \eta_c' \circ \mathrm{pos}_c^{w'}$ implying $(w_c', \mathrm{pos}_c^{w'})_{c\in\Sigma} \preceq (w_c'', \mathrm{pos}_c^{w''})_{c\in\Sigma}$.

Next, define $(w_c, \mathrm{pos}_c^w)_{c\in\Sigma}$ by $\mathrm{dom}\, w_c = \mathrm{dom}\, w_c'$, $\mathrm{pos}_c^w := \mathrm{pos}_c^{w'}$ and

$$w_c(i) := \begin{cases} x_c(i) & \text{if } i \in \mathrm{dom}\, x_c \\ w_c'(i) & \text{otherwise.} \end{cases}$$

Again, since $\operatorname{im}\operatorname{pos}_c^w = \operatorname{im}\operatorname{pos}_c^{w'} \subseteq \operatorname{dom} w'_c \cup \{0\} = \operatorname{dom} w_c \cup \{0\}$, the tuple $(w_c, \operatorname{pos}_c^w)_{c\in\Sigma}$ belongs to S. Furthermore $|\operatorname{dom} w_c| = |\operatorname{dom} w'_c| \leq |\operatorname{dom} x_c| + |\operatorname{im}\operatorname{pos}_c^{w''}| + |\operatorname{im}\operatorname{pos}_c^{v''}| + 1$ implies $|w_c| - |x_c| \leq 2|\Sigma| + 1$. Thus, the first statement holds.

Note that $\operatorname{dom} x_c \subseteq \operatorname{dom} w'_c = \operatorname{dom} w_c$. Furthermore, we showed above $\sup\operatorname{dom} x_c = \sup\operatorname{dom} w'_c$; hence $\sup\operatorname{dom} x_c = \sup\operatorname{dom} w_c$. Finally, for $i \in \operatorname{dom} x_c$, we have $x_c(i) = w_c(i)$. Thus, the identity function $\operatorname{dom} x_c \cup \{0\} \to \operatorname{dom} w_c \cup \{0\}$ is an embedding of x_c into w_c. Since, in addition, $\operatorname{pos}_c^w = \operatorname{pos}_c^{w'} = \operatorname{pos}_c^x$, we get $(x_c, \operatorname{pos}_c^x)_{c\in\Sigma} \preceq (w_c, \operatorname{pos}_c^w)_{c\in\Sigma}$.

For $i \in \operatorname{dom} x_c$, we have $w_c(i) = x_c(i) \sqsubseteq_c w''_c(i) = w'_c(i)$. Now $(w_c, \operatorname{pos}_c^w)_{c\in\Sigma} \sqsubseteq (w'_c, \operatorname{pos}_c^{w'})_{c\in\Sigma}$ follows immediately since $w_c(i) = w'_c(i)$ for $i \in \operatorname{dom} w'_c \setminus \operatorname{dom} x_c$, $\operatorname{dom} w_c = \operatorname{dom} w'_c$ and $\operatorname{pos}_c^w = \operatorname{pos}_c^{w'}$.

Finally, we construct $(v'_c, \operatorname{pos}_c^{v'})_{c\in\Sigma}$: Let $\operatorname{dom} v'_c := \operatorname{dom} w'_c \cap \operatorname{dom} v''_c$, and define $v'_c := w'_c \restriction \operatorname{dom} v'_c$ and $\operatorname{pos}_c^{v'} := \operatorname{pos}_c^{v''}$ for $c \in \Sigma$. Then $(v'_c, \operatorname{pos}_c^{v'})_{c\in\Sigma} \in S$ since $\operatorname{im}\operatorname{pos}_c^{v'} = \operatorname{im}\operatorname{pos}_c^{v''} \subseteq (\operatorname{dom} w'_c \cap \operatorname{dom} v''_c) \cup \{0\} = \operatorname{dom} v'_c \cup \{0\}$. For $i \in \operatorname{dom} v'_c$, we have $v'_c(i) = w'_c(i) = w''_c(i)$ by the definition of v'_c and of w'_c, respectively. In addition, $i \in \operatorname{dom} v''_c$ and, from $(v''_c, \operatorname{pos}_c^{v''})_{c\in\Sigma} \to (w''_c, \operatorname{pos}_c^{w''})_{c\in\Sigma}$, we obtain $w'_c(i) = v''_c(i)$ i.e. we showed $v'_c(i) = v''_c(i)$.

Now let $c \neq a$. Above, we showed $\sup\operatorname{dom} w'_c = \sup\operatorname{dom} w''_c$. We infer $v''_c = w''_c$ from $(v''_c, \operatorname{pos}_c^{v''})_{c\in\Sigma} \to (w''_c, \operatorname{pos}_c^{w''})_{c\in\Sigma}$. Therefore $\sup\operatorname{dom} w'_c = \sup\operatorname{dom} v''_c$. Hence $\operatorname{dom} v'_c = \operatorname{dom} w'_c \cap \operatorname{dom} v''_c$ implies $\sup\operatorname{dom} v'_c = \sup\operatorname{dom} v''_c$. Thus, for $c \neq a$, the identity function $\operatorname{dom} v'_c \cup \{0\} \to \operatorname{dom} v''_c \cup \{0\}$ is an embedding of v'_c into v''_c. Next we show this fact for $c = a$: Since $\operatorname{dom} v'_a = \operatorname{dom} w'_a \cap \operatorname{dom} v''_a$, we obtain $\operatorname{dom} v'_a \leq \sup\operatorname{dom} v''_a$. Furthermore, $\sup\operatorname{dom} v''_a \in \operatorname{dom} w'_a \cup \{0\}$ and $\sup\operatorname{dom} \in \operatorname{dom} v''_a \cup \{0\}$ imply $\sup\operatorname{dom} v''_a \in \operatorname{dom} v'_a \cup \{0\}$. Hence $\sup\operatorname{dom} v''_a = \sup\operatorname{dom} v'_a$. Thus, indeed, the identity function $\operatorname{dom} v'_c \cup \{0\} \to \operatorname{dom} v''_c \cup \{0\}$ is an embedding of v'_c into v''_c for any $c \in \Sigma$. Since $\operatorname{pos}_c^{v''} = \operatorname{pos}_c^{v'}$, we have $(v'_c, \operatorname{pos}_c^{v'})_{c\in\Sigma} \preceq (v''_c, \operatorname{pos}_c^{v''})_{c\in\Sigma}$ as required.

It remains to show $(v'_c, \operatorname{pos}_c^{v'})_{c\in\Sigma} \to (w'_c, \operatorname{pos}_c^{w'})_{c\in\Sigma}$, i.e. that (ii)-(iv) hold for the states $(v'_c, \operatorname{pos}_c^{v'})_{c\in\Sigma}$ and $(w'_c, \operatorname{pos}_c^{w'})_{c\in\Sigma}$ and for a, J, p_b for $b \in J$ and q:
(ii) For $c \neq a$ we have $\operatorname{dom} v'_c = \operatorname{dom} w'_c \cap \operatorname{dom} v''_c = \operatorname{dom} w'_c \cap \operatorname{dom} w''_c$ since $\operatorname{dom} v''_c = \operatorname{dom} w''_c$ follows from $(v''_c, \operatorname{pos}_c^{v''})_{c\in\Sigma} \to (w''_c, \operatorname{pos}_c^{w''})_{c\in\Sigma}$. Now $\operatorname{dom} w'_c \subseteq \operatorname{dom} w''_c$ implies $\operatorname{dom} v'_c = \operatorname{dom} w'_c$. Thus $v'_c = w'_c \restriction \operatorname{dom} v'_c = w'_c$. Similarly, we obtain $\operatorname{dom} v'_a = \operatorname{dom} w'_a \cap \operatorname{dom} v''_a = \operatorname{dom} w'_a \cap (\operatorname{dom} w''_a \setminus \{\sup\operatorname{dom} w''_a\})$. Recall that $\sup\operatorname{dom} w'_a = \sup\operatorname{dom} w''_a$ and therefore $\operatorname{dom} v'_a = \operatorname{dom} w'_a \setminus \{\sup\operatorname{dom} w'_a\}$. Since $w'_a(\sup\operatorname{dom} w'_a) = q$, we obtain $w'_a = v'_a q$ from $v'_a = w'_a \restriction \operatorname{dom} v'_a$.
(iii) Let $b, c \in \Sigma$ with $c \notin J$ or $a \neq b$. Then $\operatorname{pos}_c^{v'}(b) = \operatorname{pos}_c^{v''}(b)$ and $\operatorname{pos}_c^{w''}(b) = \operatorname{pos}_c^{w'}(b)$. Using (iii) for the states $(v''_c, \operatorname{pos}_c^{v''})_{c\in\Sigma}$ and $(w''_c, \operatorname{pos}_c^{w''})_{c\in\Sigma}$, we obtain $\operatorname{pos}_c^{v''}(b) = \operatorname{pos}_c^{w''}(b)$ and therefore $\operatorname{pos}_c^{v'}(b) = \operatorname{pos}_c^{w'}(b)$ as required.
(iv) Let $c \in J$. Since (iv) holds for the states $(v''_c, \operatorname{pos}_c^{v''})_{c\in\Sigma}$ and $(w''_c, \operatorname{pos}_c^{w''})_{c\in\Sigma}$, we get $\operatorname{pos}_c^{v'}(a) = \operatorname{pos}_c^{v''}(a) < \operatorname{pos}_c^{w''}(a) = \operatorname{pos}_c^{w'}(a)$ and $\operatorname{pos}_c^{w''}(a) \in \operatorname{dom} v''_c$. Since, in addition, $\operatorname{pos}_c^{w'}(a) \in \operatorname{dom} w'_c \cup \{0\}$, we infer $\operatorname{pos}_c^{w'}(a) \in \operatorname{dom} v''_c \cap (\operatorname{dom} w'_c \cup \{0\}) = \operatorname{dom} v''_c \cap \operatorname{dom} w'_c = \operatorname{dom} v'_c$. Finally, we get $v'_c \circ \operatorname{pos}_c^{w'}(a) = v''_c \circ \operatorname{pos} w''_c(a) = p_c$. $\square$

The two preceding lemmas are the basis for our proof that a finite basis of $\mathrm{Pred}(\uparrow(x_c, \mathrm{pos}_c^x)_{c\in\Sigma})$ can be computed for any state $(x_c, \mathrm{pos}_c^x)_{c\in\Sigma}$:

Lemma 3.2.8 *There exists an algorithm that computes the following function:*
input: *1. an alphabet* Σ,
2. a basis algorithm of an effective and monotone Σ*-ACM* $\mathcal{A}$,
3. a finite basis B_c *of* $(Q_c, \sqsubseteq_c)$ *and an algorithm to decide* $\sqsubseteq_c$ *for* $c \in \Sigma$, *and*
4. a state $(x_c, \mathrm{pos}_c^x)_{c\in\Sigma} \in S$
output: *a finite basis of the set* $\mathrm{Pred}(\uparrow(x_c, \mathrm{pos}_c^x)_{c\in\Sigma})$.

Proof. For simplicity, let M denote the set $\mathrm{Pred}(\uparrow(x_c, \mathrm{pos}_c^x)_{c\in\Sigma})$. Let H be the finite set of all states $(w_c, \mathrm{pos}_c^w)_{c\in\Sigma}$ in S that satisfy
$\mathrm{dom}\, x_c \subseteq \mathrm{dom}\, w_c$,
$w_c(i) = x_c(i)$ for $i \in \mathrm{dom}\, x_c$ and $w_c(i) \in B_c$ otherwise,
$\mathrm{pos}_c^x = \mathrm{pos}_c^w$ and
$|w_c| - |x_c| \leq 2|\Sigma| + 1$ for $c \in \Sigma$.
Note that H can be computed effectively. Furthermore, the identity functions witness $(x_c, \mathrm{pos}_c^x)_{c\in\Sigma} \preceq (w_c, \mathrm{pos}_c^w)_{c\in\Sigma}$ for $(w_c, \mathrm{pos}_c^w)_{c\in\Sigma} \in H$.

For $(w_c, \mathrm{pos}_c^w)_{c\in\Sigma} \in H$, by Lemma 3.2.6, we can compute a finite basis $B((w_c, \mathrm{pos}_c^w)_{c\in\Sigma})$ (with respect to $\sqsubseteq$) of the set of all states $(v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma}$ satisfying

$$\exists (w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma} \in S: \quad \begin{array}{ccc} (v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma} & \to & (w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma} \\ & & \sqcup\!\!\mid \\ & & (w_c, \mathrm{pos}_c^w)_{c\in\Sigma}. \end{array}$$

Let $(v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma} \in B((w_c, \mathrm{pos}_c^w)_{c\in\Sigma})$. Then there exists a state $(w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma}$ that is a successor of $(v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma}$ and dominates $(w_c, \mathrm{pos}_c^w)_{c\in\Sigma}$ with respect to $\sqsubseteq$. Since $(x_c, \mathrm{pos}_c^x)_{c\in\Sigma} \preceq (w_c, \mathrm{pos}_c^w)_{c\in\Sigma}$, we therefore get $(x_c, \mathrm{pos}_c^x)_{c\in\Sigma} \preceq (w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma}$. But this implies $(v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma} \in \mathrm{Pred}(\uparrow(x_c, \mathrm{pos}_c^x)_{c\in\Sigma})$, i.e. we showed $B((w_c, \mathrm{pos}_c^w)_{c\in\Sigma}) \subseteq \mathrm{Pred}(\uparrow(x_c, \mathrm{pos}_c^x)_{c\in\Sigma}) = M$. Now define

$$B := \bigcup_{(w_c, \mathrm{pos}_c^w)_{c\in\Sigma} \in H} B((w_c, \mathrm{pos}_c^w)_{c\in\Sigma}).$$

It remains to show that B is a basis of M: So let $(v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma} \in B$. Then there exist $(w_c, \mathrm{pos}_c^w)_{c\in\Sigma} \in H$ and $(w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma} \in S$ such that

$$\begin{array}{ccc} (v'_c, \mathrm{pos}_c^{v'})_{c\in\Sigma} & \to & (w'_c, \mathrm{pos}_c^{w'})_{c\in\Sigma} \\ & & \sqcup\!\!\mid \\ & & (w_c, \mathrm{pos}_c^w)_{c\in\Sigma} \\ & & \curlyvee\!\mid \\ & & (x_c, \mathrm{pos}_c^x)_{c\in\Sigma}. \end{array}$$

Hence $(w'_c, \mathrm{pos}^{w'}_c)_{c\in\Sigma} \succeq (x_c, \mathrm{pos}^x_c)_{c\in\Sigma}$ and therefore $(v'_c, \mathrm{pos}^{v'}_c)_{c\in\Sigma} \in M$, i.e. we showed $B \subseteq M$ which implies $\uparrow B \subseteq \uparrow M$.

Now let $(v''_c, \mathrm{pos}^{v''}_c)_{c\in\Sigma} \in M$. Then there exists $(w''_c, \mathrm{pos}^{w''}_c)_{c\in\Sigma} \in S$ such that $(v''_c, \mathrm{pos}^{v''}_c)_{c\in\Sigma} \to (w''_c, \mathrm{pos}^{w''}_c)_{c\in\Sigma} \succeq (x_c, \mathrm{pos}^x_c)_{c\in\Sigma}$. Hence, by Lemma 3.2.7, there are $(w_c, \mathrm{pos}^w_c)_{c\in\Sigma} \in H$, $(v'_c, \mathrm{pos}^{v'}_c)_{c\in\Sigma} \in B$ and $(w'_c, \mathrm{pos}^{w'}_c)_{c\in\Sigma} \in S$ with

$$\begin{array}{ccc}
(v''_c, \mathrm{pos}^{v''}_c)_{c\in\Sigma} & \to & (w''_c, \mathrm{pos}^{w''}_c)_{c\in\Sigma} \\
\curlyvee| & & \curlyvee| \\
(v'_c, \mathrm{pos}^{v'}_c)_{c\in\Sigma} & \to & (w'_c, \mathrm{pos}^{w'}_c)_{c\in\Sigma} \\
 & & \sqcup| \\
 & & (w_c, \mathrm{pos}^{w}_c)_{c\in\Sigma} \\
 & & \curlyvee| \\
 & & (x_c, \mathrm{pos}^{x}_c)_{c\in\Sigma}.
\end{array}$$

Hence $v'' \in \uparrow B$ and therefore $M \subseteq \uparrow B$. Since this trivially implies $\uparrow M \subseteq \uparrow B$, the set B is indeed a finite basis of M. □

The results we obtained so far in this section can be summarized as follows: From a monotone and effective Σ-ACM, we defined a WSTS. This WSTS has a decidable wqo and a computable finite predecessor basis. In other words, we can apply Theorem 3.2.1 and even Theorem 3.2.2. Hence we can decide whether there is a state in the WSTS that is reachable from $(v_c, \mathrm{pos}^v_c)_{c\in\Sigma}$ and dominates $(w_c, \mathrm{pos}^w_c)_{c\in\Sigma}$ for any states $(v_c, \mathrm{pos}^v_c)_{c\in\Sigma}$ and $(w_c, \mathrm{pos}^w_c)_{c\in\Sigma}$. It remains to relate this decidability to the Σ-ACM we started with.

3.3 The emptiness is decidable for ACMs

To apply the decidability result of Finkel & Schnoebelen (Theorem 3.2.2) to Σ-ACMs, we have to relate runs of a Σ-ACM and paths in the transition system $(S, \to)$. Roughly speaking, states of the form $\mathrm{state}(r)$ for some run r correspond to reachable states in $(S, \to)$. Here, "reachable" means reachable from a depth-1-state defined as follows:

Let $\mathcal{A}$ be some Σ-ACM and $\mathcal{S}(\mathcal{A}) = (S, \to, \preceq)$ be the associated WSTS. A state $(w_a, \mathrm{pos}^w_a)_{a\in\Sigma} \in S$ is a *depth-1-state* if

1. $|w_a| \leq 1$ for $a \in \Sigma$,
2. $\mathrm{pos}^w_a(b) = 0$ for $a, b \in \Sigma$, and
3. $w_a(\min \mathrm{dom}(w_a)) \in \delta_{a,\emptyset}$ for $a \in \Sigma$ with $w_a \neq \varepsilon$.

Let $(w_a, \mathrm{pos}^w_a)_{a\in\Sigma}$ be some depth-1-state. Let $V = \{a \in \Sigma \mid w_a \neq \emptyset\}$ and $E = \emptyset$. Finally, let $\lambda = \mathrm{id}_V$. Since $w_a(\min \mathrm{dom}(w_a)) \in \delta_{a,\emptyset}$ for $a \in V$, the mapping $a \mapsto w_a(\min \mathrm{dom}(w_a))$ is a run of $\mathcal{A}$ on the Σ-dag $t = (V, E, \lambda)$. Furthermore, the

Σ-dag t is (considered as a partial order) an antichain since $E = \emptyset$. If conversely t is an antichain and r is a run of $\mathcal{A}$ on t, then state(r) is a depth-1-state.

Unfortunately, the truth is not that simple, i.e., there are states reachable from a depth-1-state, that are not of the form state(r) for some run r:

Example 3.3.1 Let $\Sigma = \{a, b\}$, $Q_a = \{q_a\}$ and $Q_b = \{q_b\}$. Furthermore, the transitions of the asynchronous-cellular automaton $\mathcal{A}$ are given by $\{q_a\} = \delta_{a,\{b\}}(q_b) = \delta_{a,\emptyset}$, $\{q_b\} = \delta_{b,\emptyset}$, and $\emptyset = \delta_{c,M}$ otherwise. Then $v_a = q_a$ and $v_b = q_b$ define a depth-1-state. Now let $w_a = q_a q_a$, $w_b = v_b$, $\text{pos}_a^w = \{(a, 0), (b, 0)\}$ and $\text{pos}_b^w = \{(a, 1), (b, 0)\}$. These entities define a state $(w_c, \text{pos}_c^w)_{c \in \Sigma}$ that is a successor state of the depth-1-state $(v_c, \text{pos}_c^v)_{c \in \Sigma}$. But there is no Σ-dag t and run r of $\mathcal{A}$ on t such that $(w_c, \text{pos}_c^w)_{c \in \Sigma} = \text{state}(r)$.

To make the idea that a state is reachable iff it corresponds to a run of the ACM $\mathcal{A}$ work, we will define finitely many asynchronous-cellular automata $\mathcal{A}(f)$ for $f : \Sigma \to \Sigma$ with the following nice properties:

- A Σ-dag t is accepted by $\mathcal{A}$ iff there exists a function $f : \Sigma \to \Sigma$ such that t is accepted by $\mathcal{A} \times \mathcal{A}(f)$.
- In the well-structured transition system associated to $\mathcal{A} \times \mathcal{A}(f)$, reachable states correspond to runs of $\mathcal{A} \times \mathcal{A}(f)$.

Then, for any function f, we can apply Finkel & Schnoeblen's result on the associated well-structured transition system, and combine the outcomes into an answer whether $L(\mathcal{A})$ is empty.

After explaining the idea of our proof, we now come to the technicalities: Let Σ be an alphabet. A *weak Σ-dag* is a triple (V, E, λ) where (V, E) is a finite directed acyclic graph and $\lambda : V \to \Sigma$ is a labeling function such that

1. for all $x, y \in \min(V, E^\star)$ with $\lambda(x) = \lambda(y)$, we have $x = y$, and
2. for any $(x, y), (x', y') \in E$ with $\lambda(x) = \lambda(x')$, $\lambda(y) = \lambda(y')$, we have $x = x'$ if and only if $y = y'$.

Note that any Σ-dag is a weak Σ-dag. Similarly to Σ-dags, we can define $\mathrm{R}(y)$ for a node y in a weak Σ-dag (V, E, λ) to be the set of all labels $\lambda(x)$ with $(x, y) \in E$. Since in a weak Σ-dag for any node y and any $a \in \mathrm{R}(y)$ there is a unique node x with $\lambda(x) = a$ and $(x, y) \in E$, we can also use the notion $\partial_a(y)$ to denote this vertex. Hence, for a Σ-ACM $\mathcal{A}$, we can speak of a mapping $r : V \to Q$ that satisfies the run condition at a node $x \in V$ relative to t.

Lemma 3.3.2 *There exists an algorithm that on input of an alphabet Σ and a function $f : \Sigma \to \Sigma$ outputs an asynchronous cellular automaton $\mathcal{A}(f)$ such that*

1. $\bigcup_{f \in \Sigma^\Sigma} L(\mathcal{A}(f)) = \mathbb{D}$, *and*
2. *for any weak Σ-dag $t = (V, E, \lambda)$, any $f : \Sigma \to \Sigma$ and any mapping r that satisfies the run condition of $\mathcal{A}(f)$ for any $x \in V$ relative to t, the set $\lambda^{-1}(a)$ is a chain w.r.t. $E^\star$ for any $a \in \Sigma$.*

Proof. First, we give the construction of the ACAs $\mathcal{A}(f)$: Let $f : \Sigma \to \Sigma$. The set of local states shared by all processes equals the set of nonempty partial functions from Σ to itself, i.e. $Q = Q_a = \text{part}(\Sigma, \Sigma)$ for $a \in \Sigma$. The transition functions $\delta_{a,J}$ are defined by

$$\delta_{a,\emptyset} = \{g \in \text{part}(\Sigma, \Sigma) \mid a \in \text{dom}(g) = f^{-1}(a) \neq \emptyset\}$$

and for $J \neq \emptyset$ by

$$g \in \delta_{a,J}((g_b)_{b \in J}) \iff a \in \text{dom}(g) \text{ and } (\forall c \in \text{dom}(g) \exists b \in J : g_b(c) = a)$$

for $g_b \in \text{part}(\Sigma, \Sigma)$ for $b \in J$. Finally, all tuples of local states are accepting.

To show the first statement, let $t = (V, E, \lambda) \in \mathbb{D}$ be a Σ-dag. Since t is a Σ-dag, nodes that carry the same label are linearly ordered with respect to $E^\star$. Hence, we can choose maximal chains $C_a \subseteq V$ with $\lambda^{-1}(a) \subseteq C_a$ for any $a \in \Sigma$. Note that the minimal node of the chain C_a is minimal in t. We set $f(a) := \lambda(\min C_a)$ and obtain a function $f : \Sigma \to \Sigma$. To prove the first statement, it remains to show that $\mathcal{A}(f)$ accepts t: We define a mapping $r : V \to Q = \text{part}(\Sigma, \Sigma)$ with $\text{dom}(r(x)) = \{a \in \Sigma \mid x \in C_a\}$. Now let $x \in V$ and $a \in \text{dom}(r(x))$. If there exists $y \in C_a$ with $(x, y) \in E$, then there exists a least such node y since C_a is a chain. Let $r(x)(a)$ be the label of this minimal node. If there is no such node y, define $r(x)(a) := a$. Since $x \in C_{\lambda(x)}$ for any $x \in V$, the function $r(x)$ is indeed nonempty and therefore belongs to Q. Now let $y \in V$ be some node with $a = \lambda(y)$. We want to show that r satisfies the run condition of $\mathcal{A}(f)$ at y relative to t: First let y be minimal in t. Since $\lambda^{-1}(a) \subseteq C_a$, we get $a \in \text{dom}(r(y))$. Now let $b \in f^{-1}(a)$, i.e. $f(b) = a$. Then by the choice of f, we get $a = \lambda(\min C_b)$. Since C_b is a maximal chain, the node $\min C_b$ is minimal in t. Since t is a Σ-dag, its minimal nodes carry mutually different labels. Hence $y = \min C_b \in C_b$. This implies $b \in \text{dom}(r(y))$ and therefore $f^{-1}(a) \subseteq \text{dom}(r(y))$. Conversely let $b \in \text{dom}(r(y))$. Then $y \in C_b$ and, since y is minimal in t, $y = \min C_b$. Hence $a = \lambda(y) = \lambda(\min C_b) = f(b)$ ensures $\text{dom}(r(y)) \subseteq f^{-1}(a)$. Thus, the mapping r satisfies the run condition of $\mathcal{A}(f)$ at the minimal nodes of t relative to t. Now let $y \in V$ be nonminimal. Then $J := \text{R}(y) \neq \emptyset$. Since $y \in \lambda^{-1}(a) \subseteq C_a$, we get $a \in \text{dom}(r(y))$. Now let $c \in \text{dom}(r(y))$, i.e. $y \in C_c$. Since C_c is a maximal chain, there exists a lower neighbor (with respect to the partial order $E^\star$) x of y which belongs to the chain C_c. Hence $(x, y) \in E$ and $c \in \text{dom}(r(x))$. Furthermore, x is not maximal in $(V, E^\star)$. Let $y' \in C_c$ be minimal with $(x, y') \in E$. Then $\lambda(y') = r(x)(c)$. Since $(x, y) \in E^+$ and $y \in C_c$, we obtain $xE^+y'E^\star y$ which ensures $y' = y$. Hence $\lambda(y) = r(x)(c)$.

Now we prove the second statement of the lemma. Let $f : \Sigma \to \Sigma$ be some mapping. Furthermore let $t = (V, E, \lambda)$ be a weak Σ-dag and let $r : V \to Q$ be a mapping that satisfies the run condition of $\mathcal{A}(f)$ for any node $x \in V$ relative to t. We will prove that $C_a := \{x \in V \mid a \in \text{dom}(r(x))\}$ is a chain. Since by the definition of the transition functions $\delta_{a,J}$ we have $\lambda(x) \in \text{dom}(r(x))$ for any $x \in V$, this will imply $\lambda^{-1}(a) \subseteq C_a$ and therefore that $\lambda^{-1}(a)$ is linearly ordered.

Now let $x, y \in C_c$. Since r satisfies the run condition of $\mathcal{A}(f)$, there exist $x_0, x_1, \ldots, x_n \in V$ such that $x_0 \in \min(t)$, $x_n = x$, $(x_i, x_{i+1}) \in E$ for $0 \leq i < n$, and $c \in \text{dom}(r(x_i))$ and $r(x_i)(c) = \lambda(x_{i+1})$ for $0 \leq i < n$.

Similarly, we find nodes $y_0, y_1, \ldots, y_m \in V$ such that $y_0 \in \min(t)$, $y_m = y$, $(y_i, y_{i+1}) \in E$ for $0 \leq i < m$, and $c \in \text{dom}(r(y_i))$ and $r(y_i)(c) = \lambda(y_{i+1})$ for $0 \leq i < m$. Without loss of generality, we may assume $n \leq m$.

Since $c \in \text{dom}(r(x_0))$ and $\text{R}(x_0) = \emptyset$, we obtain $c \in f^{-1}(\lambda(x_0))$ since the run condition is satisfied at the node x_0. Hence $f(c) = \lambda(x_0)$ and similarly $f(c) = \lambda(y_0)$. Since the minimal nodes of the weak Σ-dag t carry different labels, this implies $x_0 = y_0$. By induction, let $0 \leq i < n$ with $x_i = y_i$. Then $(x_i, x_{i+1}) \in E$, $(y_i, y_{i+1}) \in E$ and $\lambda(x_{i+1}) = r(x_i)(c) = r(y_i)(c) = \lambda(y_{i+1})$. Since t is a weak Σ-dag, this implies $x_{i+1} = y_{i+1}$. Thus, we get $x = x_n = y_n E^\star y$ as required. □

Now let $\mathcal{A}_i = ((Q_a^i, \sqsubseteq_a^i)_{a\in\Sigma}, (\delta_{a,J}^i)_{a\in\Sigma, J\subseteq\Sigma}, F^i)$ for $i = 1, 2$ be two Σ-ACMs. Then the direct product $\mathcal{A}_1 \times \mathcal{A}_2 = ((Q_a, \sqsubseteq_a)_{a\in\Sigma}, (\delta_{a,J})_{a\in\Sigma, J\subseteq\Sigma}, F)$ has the following obvious definition:

$$\begin{aligned}
Q_a &:= Q_a^1 \times Q_a^2, \\
\sqsubseteq_a &:= \sqsubseteq_a^1 \times \sqsubseteq_a^2 \\
\delta_{(a,J)}((p_b^1, p_b^2)_{b\in J}) &:= \delta_{(a,J)}^1((p_b^1)_{b\in J}) \times \delta_{(a,J)}^2((p_b^2)_{b\in J}), \text{ and} \\
F &:= \{(q_a^1, q_a^2)_{a\in J} \mid (q_a^i)_{a\in J} \in F^i \text{ for } i = 1, 2\}.
\end{aligned}$$

It is easily seen that the direct product of monotone and effective ACMs is monotone and effective, again. Furthermore, this direct product accepts the intersection of the two languages, i.e. $L(\mathcal{A}_1 \times \mathcal{A}_2) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$. Hence, to decide whether $L(\mathcal{A})$ is empty, by the first statement of the preceding lemma, it suffices to decide whether $L(\mathcal{A} \times \mathcal{A}(f))$ is empty for each function $f : \Sigma \to \Sigma$. This is the reason why we now start to consider these direct products.

Lemma 3.3.3 *Let $\mathcal{A}'$ be a Σ-ACM and $f : \Sigma \to \Sigma$. Let $\mathcal{A} = \mathcal{A}' \times \mathcal{A}(f)$ and let $\mathcal{S}(\mathcal{A}) = (S, \to, \preceq)$ be the WSTS associated with $\mathcal{A}$. Let $(w_a, \text{pos}_a^w)_{a\in\Sigma}$ be a state of $(S, \to)$. Then the following are equivalent:*

(1) There exist a Σ-dag t and a run r of the Σ-ACM $\mathcal{A}$ on t such that $\text{state}(r) = (w_a, \text{pos}_a^w)_{a\in\Sigma}$.

(2) The state $(w_a, \text{pos}_a^w)_{a\in\Sigma}$ is reachable from some depth-1-state in the transition system $(S, \to)$.

Proof. Throughout this proof, let $\mathcal{A} = ((Q_a, \sqsubseteq_a)_{a\in\Sigma}, (\delta_{a,J})_{a\in\Sigma, J\subseteq\Sigma}, F)$.

(1) $\to$ (2): Let $t = (V, E, \lambda)$ be a Σ-dag and $r : V \to Q$ a run of $\mathcal{A}$ on t. Recall that we assume $V \subseteq \mathbb{N}^+$ with $x < y$ whenever $(x, y) \in E$. We can in addition require that $x \in \min(V, E^\star)$ and $y \in V \setminus \min(V, E^\star)$ imply $x < y$. Since the linear order $\leq$ of natural numbers extends the partial order $E^\star$ on V, we can enumerate V such that $V = \{x_1, x_2, \ldots, x_n\}$ with $x_i < x_{i+1}$. Furthermore, there is $k \in \mathbb{N}^+$ such that $\min(V, E^\star) = \{x_1, x_2, \ldots, x_k\}$ by our additional requirement. For $i = k, k+1, k+2, \ldots, n$, let $V_i := \{x_1, x_2, \ldots, x_i\}$, $t_i := (V_i, E, \lambda)$ and $r_i : V_i \to Q$ be the restriction of r to V_i. Then, for all suitable i, r_i is a run of the Σ-ACM $\mathcal{A}$ on the Σ-dag t_i. Furthermore, $V_k = \{x_1, x_2, \ldots, x_k\}$ is the set of minimal nodes of t with respect to $E^\star$ and t_k is the restriction of t to its minimal nodes. Hence $\text{state}(r_k)$ is a depth-1-state. It remains to prove $\text{state}(r_i) \to \text{state}(r_{i+1})$ for $k \leq i < n$ to obtain the desired result by induction. Let $(v_a, \text{pos}^v_a)_{a\in\Sigma} = \text{state}(r_i)$ and $(w_a, \text{pos}^w_a)_{a\in\Sigma} = \text{state}(r_{i+1})$. Furthermore, let $a = \lambda(x_{i+1})$, $J = \text{R}(x_{i+1})$, $p_b = r\partial_b(x_{i+1})$ for $b \in J$, and $q = r(x_{i+1})$.

We show that (i)-(iv) hold for these elements: Since $i+1 > k$, the node x_{i+1} is not minimal in t. Hence it is the target of some edge from E, i.e. $J \neq \emptyset$. Since r_{i+1} is a run on t_{i+1}, we get $q \in \delta_{a,J}((p_b)_{b\in J})$ and therefore (i). Since $V_{i+1} \setminus V_i = \{x_{i+1}\}$ and $r_i = r_{i+1} \restriction V_i$, we get $r_i \restriction \lambda^{-1}(c) = r_{i+1} \restriction \lambda^{-1}(c)$ for $c \neq a$. Hence $w_c = v_c$ for $c \neq a$. Furthermore, $w_a = r_{i+1} \restriction \lambda^{-1}(a) = (r_i \restriction \lambda^{-1}(a))\, r_{i+1}(x_{i+1}) = v_a q$. Thus, we showed (ii). Note that the only edges in t_{i+1} that do not belong to t_i are of the form (x, x_{i+1}) with $\lambda(x) \in \text{R}(x_{i+1}) = J$, i.e. their source is labeled by an element of J while the target is labeled by a. Hence, for $b, c \in \Sigma$ with $c \in J$ or $a \neq b$, we have $\text{pos}^v_c(b) = \text{pos}^w_c(b)$, i.e. (iii) holds.

To show (iv), let $c \in J$. Then $\text{pos}^w_c(a) = \partial_c(x_{i+1})$ since there is an edge (x, x_{i+1}) in t_{i+1} with $\lambda(x) = c$. Let $y, z \in V_i$ such that $\lambda(y) = a$ and $\lambda(z) = b$ with $(z, y) \in E$. Then $y < x_{i+1}$ and therefore $z < \partial_c(x_{i+1})$ by the second requirement on Σ-dags. Hence $\text{pos}^v_c(a) < \text{pos}^w_c(a)$. Since $\partial_c(x_{i+1}) \in V_i$, we also get $\text{pos}^w_c(a) \in \lambda^{-1}(c) \cap V_i = \operatorname{dom} v_c$. By the very definition of p_c, we have $w_c \circ \text{pos}^w_c(a) = r \circ \text{pos}^w_c(a) = r\partial_c(x_{i+1}) = p_c$, i.e. (iv) holds. Thus we showed $\text{state}(r_i) \to \text{state}(r_{i+1})$ and therefore the implication (1) $\to$ (2).

(2) $\to$ (1): When we defined the concept of a depth-1-state, we showed that they are of the form $\text{state}(r)$ for some run r of $\mathcal{A}$. Hence the implication (2) $\to$ (1) holds for depth-1-states and it remains to show that, given a run r, any successor of $\text{state}(r)$ in $(S, \to)$ is of the form $\text{state}(r')$ for some run r' of the Σ-ACM $\mathcal{A}$. So let $t = (V, E, \lambda)$ be a Σ-dag and let $r : V \to Q$ be a run of $\mathcal{A}$ on t. Furthermore, let $\text{state}(r) = (v_a, \text{pos}^v_a)_{a\in\Sigma} \to (w_a, \text{pos}^w_a)_{a\in\Sigma}$. Then there exist $a \in \Sigma$, $\emptyset \neq J \subseteq \Sigma$, $p_b \in Q_b$ for $b \in J$ and $q \in Q_a$ such that (i)-(iv) hold. Define $V' := V \,\dot\cup\, \{z\}$ and let $\lambda' := \lambda \cup \{(z, a)\}$. The set of edges E' will consist of all edges from E and some edges of the form (x, z) with $x \in V$. According to the definition of a run, we should have additional edges with $\lambda(x) \in J$ only and, conversely, for any $c \in J$ there has to be a new edge (x, z) with $\lambda(x) = c$. Furthermore, the state at the source of this new edge should equal p_c. By (iv), $\text{pos}^w_c(a) \in \operatorname{dom} v_c = \lambda^{-1}(c)$.

Hence $\mathrm{pos}_c^w(a)$ belongs to V and is labeled by c. Now we define

$$E' := E \mathbin{\dot\cup} \{(\mathrm{pos}_c^w(a), z) \mid c \in J\}.$$

Then (V', E') is a dag since the only new edges have a common target z. We show that $t' = (V', E', \lambda')$ is a weak Σ-dag: Since $J \neq \emptyset$, there is an edge whose target is z, i.e. z is not minimal in $(V', (E')^\star)$. In other words $\min(t) = \min(t')$. Since t is a Σ-dag, this implies that the minimal nodes of t' carry mutually different labels as required by the first axiom for weak Σ-dags. Now let $(x, y), (x', y') \in E'$ with $\lambda(x) = \lambda(x')$ and $\lambda(y) = \lambda(y')$. We have to show $x = x' \iff y = y'$. Since t is a Σ-dag, this holds if $(x, y), (x', y') \in E$. So assume $(x, y) \in E' \setminus E$, i.e. $y = z$ and $x = \mathrm{pos}_c^w(a)$ for some $c \in J$. If $x = x'$, we get $x' = x = \mathrm{pos}_c^w(a) > \mathrm{pos}_c^v(a)$ by (iv) since $c \in J$. Thus, $x' > \sup\{\overline{x} \in \lambda^{-1}(c) \mid \exists \overline{y} \in \lambda^{-1}(a) : (\overline{x}, \overline{y}) \in E\}$. Hence $(x', y') \notin E$ and therefore $y' = z = y$. Conversely assume $y = y'$. Then $x = \mathrm{pos}_{\lambda(x)}^w(a) = \mathrm{pos}_{\lambda(x')}^w(a) = x'$. Thus, t' is indeed a weak Σ-dag.

Now let $r' := r \mathbin{\dot\cup} \{(z, q)\}$. For $x \in V$, this mapping satisfies the run condition of $\mathcal{A}$ relative to t and therefore relative to t'. Since the edges in t' with target z are of the form $(\mathrm{pos}_c^w(a), z)$ with $c \in J$, we have $\mathrm{R}(z) = J$ and $\partial_c(z) = \mathrm{pos}_c^w(a)$ for $c \in J$. Hence, by (iv), $r\partial_c(z) = p_c$ for $c \in J$. Since $q \in \delta_{a,J}((p_c)_{c \in J})$, the mapping r' satisfies the run condition at z relative to t', too. Recall that $\mathcal{A}$ is the direct product of $\mathcal{A}'$ and $\mathcal{A}(f)$. Hence $\pi_2 \circ r'$ satisfies the run condition of $\mathcal{A}(f)$ at any node $x \in V'$ relative to the weak Σ-dag t'. Hence, by Lemma 3.3.2 (2), the set $(\lambda')^{-1}(b)$ is a chain w.r.t. $(E')^\star$ for any $b \in \Sigma$. To show that t' is a Σ-dag, it remains to prove the second condition, i.e. that for any $(x, y), (x', y') \in E'$ with $\lambda(x) = \lambda(x')$ and $\lambda(y) = \lambda(y')$ we have

$$(x, x') \in (E')^\star \iff (y, y') \in (E')^\star.$$

Since t is a Σ-dag, this equivalence holds if $(x, y), (x', y') \in E$.

So assume $(x', y') \in E' \setminus E$. Then $y' = z$. Since $\lambda(y) = \lambda(y')$, the nodes y and $y' = z$ are ordered w.r.t. $(E')^\star$. Since $z = y'$ is maximal in t' w.r.t. $(E')^\star$, this implies $(y, y') \in (E')^\star$. We show $(x, x') \in (E')^\star$: If $(x, y) \notin E$, we are done since then $y = y'$ and therefore $x = x'$. So assume $(x, y) \in E$. Since $(x', y') \in E' \setminus E$, there exists $c \in J$ with $x' = \mathrm{pos}_c^w(a)$. Hence $x' > \mathrm{pos}_c^v(a)$ by (iv). But $\mathrm{pos}_c^v(a) = \sup\{\overline{x} \in \lambda^{-1}(c) \mid \exists \overline{y} \in \lambda^{-1}(a) : (\overline{x}, \overline{y}) \in E\}$ and the node x is contained in this set. Hence, indeed $x' > x$ w.r.t. the linaer order on the natural numbers. Since x' and x carry the same label, they are comparable w.r.t. $E^\star$. Hence $(x, x') \in E^\star$. Thus, we showed the required equivalence in case $(x', y') \in E' \setminus E$.

Now assume $(x, y) \in E' \setminus E$ and therefore $y = z$. First, let $(x, x') \in (E')^\star$ and therefore $x \leq x'$. Since $x = \mathrm{pos}_c^w(a)$ for some $c \in J$, we obtain $x \geq x'$ as above. Hence $x = x'$ and, since t' is a weak Σ-dag, $y = y'$. Thus, we showed $(y, y') \in (E')^\star$. Now assume $(y, y') \in (E')^\star$. Since $(x, y) \in E' \setminus E$, we obtain similarly to above, $(y', y) \in (E')^\star$, i.e. $y = y'$. Since t' is a weak Σ-dag, this implies $x = x'$.

Thus, t' is indeed a Σ-dag. Hence r' is a run of the Σ-ACM $\mathcal{A}$ on the Σ-dag t'. It is an easy exercise to show $\text{state}(r') = (w_a, \text{pos}_a^w)_{a\in\Sigma}$ proving the implication (2) $\rightarrow$ (1). □

Recall that we wanted to apply Theorem 3.2.4 to decide whether $L(\mathcal{A})$ is empty, i.e., eventually, the emptiness question shall be reduced to the question whether some particular state $(w_c, \text{pos}_c^w)_{c\in\Sigma}$ can be dominated by a state reachable from some state $(v_c, \text{pos}_c^v)_{c\in\Sigma}$. The results so far point in the direction that $(v_c, \text{pos}_c^v)_{c\in\Sigma}$ will be a depth-1-state. Next we define which state $(v_c, \text{pos}_c^v)_{c\in\Sigma}$ will be considered. They, of course, have to be related to the set of accepting global states of the Σ-ACM $\mathcal{A}$:

Let $\mathcal{A}' = ((Q'_a, \sqsubseteq'_a)_{a\in\Sigma}, (\delta'_{a,J})_{a\in\Sigma, J\subseteq\Sigma}, F')$ be a Σ-ACM and $f : \Sigma \rightarrow \Sigma$. Define $\mathcal{A} = \mathcal{A}' \times \mathcal{A}(f)$ and let $\mathcal{A} = ((Q_a, \sqsubseteq_a)_{a\in\Sigma}, (\delta_{a,J})_{a\in\Sigma, J\subseteq\Sigma}, F)$. Furthermore, let B_c be a finite basis of the set of local states Q_c of the product automaton $\mathcal{A} = \mathcal{A}' \times \mathcal{A}(f)$ for $c \in \Sigma$. Now let $J \subseteq \Sigma$ and let q_c be some local state from the product automaton for $c \in J$. We define $\text{States}((q_c)_{c\in J})$ to consist of all states $(w_c, \text{pos}_c^w)_{c\in\Sigma}$ from $\mathcal{S}(\mathcal{A})$ such that for all $c \in \Sigma$:

$$|w_c| \leq |\Sigma|,\ (w_c = \varepsilon \iff c \notin J), \text{ and } w_c \in B_c^\star q_c \text{ for } c \in J.$$

Note that due to the restrictions $|w_c| \leq |\Sigma|$ and $w_c \in B_c^\star q_c \cup \{\varepsilon\}$, the set $\text{States}((q_c)_{c\in J})$ is finite. Since, in addition, the set F of accepting states of $\mathcal{A}' \times \mathcal{A}(f)$ is finite, we even have that $\bigcup_{\overline{q}\in F} \text{States}(\overline{q})$ is finite. The following lemma states that $L(\mathcal{A}' \times \mathcal{A}(f))$ is not empty iff some state of this finite set $\bigcup_{\overline{q}\in F} \text{States}(\overline{q})$ is dominated by a state in $\mathcal{S}(\mathcal{A}' \times \mathcal{A}(f))$ that is reachable from a depth-1-state.

Lemma 3.3.4 *Let $\mathcal{A}'$ be a Σ-ACM, $f : \Sigma \rightarrow \Sigma$ and $\mathcal{A} = \mathcal{A}' \times \mathcal{A}(f)$. Furthermore, let $\mathcal{S}(\mathcal{A}) = (S, \rightarrow, \preceq)$. Then the following are equivalent:*

1. *$\mathcal{A}$ accepts some Σ-dag, i.e. $L(\mathcal{A}) = L(\mathcal{A}') \cap L(\mathcal{A}(f)) \neq \emptyset$.*
2. *There exist an accepting state $(q_a)_{a\in J}$ of $\mathcal{A}$, a depth-1-state $(v'_a, \text{pos}_a^{v'})_{a\in\Sigma}$ from S, a state $(w_a, \text{pos}_a^w)_{a\in\Sigma} \in \text{States}((q_a)_{a\in J})$ and a state $(w'_a, \text{pos}_a^{w'})_{a\in\Sigma}$ in S such that $(v'_a, \text{pos}_a^{v'})_{a\in\Sigma} \rightarrow^\star (w'_a, \text{pos}_a^{w'})_{a\in\Sigma} \succeq (w_a, \text{pos}_a^w)_{a\in\Sigma}$.*

Proof. Let $t = (V, E, \lambda) \in L(\mathcal{A})$. Then there exists a successful run r of $\mathcal{A}$ on t. By Lemma 3.3.3, $\text{state}(r) = (w'_a, \text{pos}_a^{w'})_{a\in\Sigma}$ is reachable from some depth-1-state $(v'_a, \text{pos}_a^{v'})_{a\in\Sigma}$. Since r is successful, there is $(q_c)_{c\in\lambda(V)} \in F$ such that $w'_c(\sup \operatorname{dom} w'_c) = r(\sup \lambda^{-1}(c)) \sqsupseteq_c q_c$ for $c \in \lambda(V) =: J$.

For $a \in \Sigma$, define a word $w_a \in B_a^\star q_a \cup \{\varepsilon\}$ as follows: Let $\operatorname{dom} w_a := (\operatorname{im} \text{pos}_a^{w'} \cup \{\sup \operatorname{dom} w'_a\}) \setminus \{0\}$. If $\operatorname{dom} w_a \neq \emptyset$, let $w_a(\max \operatorname{dom} w_a) := q_a$. For

$1 \leq i < \max \operatorname{dom} w'_a$ choose $w_a(i) \in B_a$ with $w_a(i) \sqsubseteq_a w'_a(i)$. Furthermore, let $\mathrm{pos}^w_a = \mathrm{pos}^{w'}_a$. Then $(w_a, \mathrm{pos}^w_a)_{a\in\Sigma} \preceq (w'_a, \mathrm{pos}^{w'}_a)_{a\in\Sigma}$ witnessed by the identity mapping from $\operatorname{dom} w_a \cup \{0\}$ to $\operatorname{dom} w'_a \cup \{0\}$. By the very construction it can easily be checked that $(w_c, \mathrm{pos}^w_c)_{c\in\Sigma} \in \mathrm{States}((q_c)_{c\in J}) \subseteq \bigcup_{\overline{q}\in F} \mathrm{States}(\overline{q})$.

Conversely, let $(q_a)_{a\in J} \in F$ be an accepting state of $\mathcal{A}$. Furthermore, let $(w_a, \mathrm{pos}^w_a)_{a\in\Sigma} \in \mathrm{States}((q_a)_{a\in J})$ and suppose $(v'_a, \mathrm{pos}^{v'}_a)_{a\in\Sigma} \to^\star (w'_a, \mathrm{pos}^{w'}_a)_{a\in\Sigma} \succeq (w_a, \mathrm{pos}^w_a)_{a\in\Sigma}$ for some depth-1-state $(v'_a, \mathrm{pos}^{v'}_a)_{a\in\Sigma}$. We assume furthermore that $(w'_a, \mathrm{pos}^{w'}_a)_{a\in\Sigma} \succeq (w_a, \mathrm{pos}^w_a)_{a\in\Sigma}$ is witnessed by the embeddings $\eta_a : w_a \hookrightarrow w'_a$. By Lemma 3.3.3, there exists a Σ-dag $t = (V, E, \lambda)$ and a run r of $\mathcal{A}$ on t such that $(w'_a, \mathrm{pos}^{w'}_a)_{a\in\Sigma} = \mathrm{state}(r)$. Since w'_a is the empty word iff w_a is empty, we obtain $\lambda(V) = \{a \in \Sigma \mid w'_a \neq \varepsilon\} = \{a \in \Sigma \mid w_a \neq \varepsilon\}$. For $a \in \Sigma$ with $w_a \neq \varepsilon$, i.e. for $a \in \lambda(V)$, we have $w'_a(\sup \operatorname{dom} w'_a) \sqsupseteq_a w_a(\sup \operatorname{dom} w_a) = q_a$. Hence the run r is successful. □

Summarizing the results of this section, finally we show that the emptiness of effective and monotone Σ-ACMs is uniformly decidable:

Theorem 3.3.5 *There exists an algorithm that solves the following decision problem:*

input: *1. an alphabet Σ,*
2. a basis algorithm of an effective and monotone Σ-ACM $\mathcal{A}'$,
3. the set of final states F' of $\mathcal{A}'$,
4. a finite basis of $(Q_c, \sqsubseteq_c)$, and an algorithm to decide $\sqsubseteq_c$ for $c \in \Sigma$.

output: *Is $L(\mathcal{A}')$ empty?*

Proof. We may assume that there is $a \in \Sigma$ such that $\delta_{a,\emptyset} = \{\bot\}$ and $\delta_{c,\emptyset} = \emptyset$ for $c \neq a$. Then there is only one depth-1-state $(v_c, \mathrm{pos}^v_c)_{c\in\Sigma}$.

By Lemma 3.3.2 (1), it holds $L(\mathcal{A}') = \bigcup_{f\in\Sigma^\Sigma} L(\mathcal{A}' \times \mathcal{A}(f))$. Hence it suffices to decide the emptiness of $L(\mathcal{A}' \times \mathcal{A}(f))$ for $f : \Sigma \to \Sigma$. So let $\mathcal{A} = \mathcal{A}' \times \mathcal{A}(f)$. Note that this Σ-ACM if monotone and effective, that we have access to a basis algorithm for this ACM, that we know a finite basis for the sets of local states and that we can decide the wqos of local states for any $c \in \Sigma$. Now let $\mathcal{S}(\mathcal{A}) = (S, \to, \preceq)$ be the associated transition system. By Theorem 3.2.4, it is a WSTS. It is clear that $\preceq$ is decidable using the algorithms that decide the wqos of local states in $\mathcal{A}$. By Lemma 3.2.8, from a state $(w_a, \mathrm{pos}^w_a)_{a\in\Sigma} \in S$, a finite basis of the set $\mathrm{Pred}(\uparrow(w_a, \mathrm{pos}^w_a)_{a\in\Sigma})$ can be computed effectively. Hence, by Theorem 3.2.2 the set of states that are dominated by a state reachable from $(v_a, \mathrm{pos}^v_a)_{a\in\Sigma}$ is recursive. Since $\bigcup_{\overline{q}\in F} \mathrm{States}(\overline{q})$ is finite, the result follows from Lemma 3.3.4. □

A consequence of Theorem 3.3.5 is that for any monotone and effective Σ-ACM $\mathcal{A}$ the membership in $L(\mathcal{A})$ is decidable:

Corollary 3.3.6 *Let $\mathcal{A}$ be a monotone and effective Σ-ACM. Then the set $L(\mathcal{A})$ is recursive.*

Proof. Let $t \in \mathbb{D}$ be some Σ-dag. Then one can easily construct a Σ-ACA $\mathcal{A}_t$ with $L(\mathcal{A}_t) = \{t\}$. Hence $L(\mathcal{A} \times \mathcal{A}_t)$ is empty iff $t \notin L(\mathcal{A})$. Since the emptiness of $L(\mathcal{A} \times \mathcal{A}_t)$ is decidable, so is the question "$t \in L(\mathcal{A})$?". □

Unfortunately, Theorem 3.3.5 keeps the promise made by the title of this section only partially since we have to impose additional requirements on the Σ-ACMs:

- Of course, one cannot expect that the emptiness for arbitrary Σ-ACMs is decidable. There is even a formal reason: In general, a Σ-ACM is an infinite object that has to be given in some finite form. Hence some effectiveness requirement is necessary.

- On the other hand, the monotonicity originates only in our proof using well structured transition systems. These transition systems clearly require some monotonicity but it is not clear whether this is really needed for the result on asynchronous cellular machines.

Recall that by Example 2.1.3 the set of Hasse-diagrams of all pomsets without autoconcurrency over an alphabet Σ can be accepted by some Σ-ACM. One can check that the ACM we gave is not monotone. Unfortunately, we were not able to construct a monotone Σ-ACM accepting all Hasse-diagrams nor did we succeed in showing that such a Σ-ACM does not exist. If we were able to accept all Hasse-diagrams by a monotone and effective Σ-ACM, the question "Is $L(\mathcal{A}) \cap \text{Ha}$ empty?" would be decidable for monotone and effective ACMs $\mathcal{A}$.

An asynchronous cellular *automaton* over Σ is a Σ-ACM where the sets of local states Q_c are finite for $c \in \Sigma$. Hence the identity relations on Q_c for $c \in \Sigma$ are well quasi orders. Thus, the set of Σ-ACAs $\mathcal{A}$ with $L(\mathcal{A}) \neq \emptyset$ is recursive. It is easily seen that a *deterministic* ACA can effectively be complemented. Similarly, one can effectively construct a deterministic ACA that accepts the intersection of two languages accepted by deterministic ACAs. Hence, as a consequence of the theorem above, the equivalence of deterministic Σ-ACAs is decidable. The following chapter shows that this is not the case for nondeterministic Σ-ACAs.