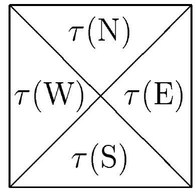
Chapter 4

The undecidability results

The result of the preceding chapter shows that one can automatically check whether a property of Σ-dags described by a Σ-ACM is contradictory. Another natural question is whether two properties are equivalent, i.e. whether two Σ-ACMs accept the same language. Since there is a Σ-ACM that accepts all Σ-dags, a special case of this equivalence problem is to ask whether a given Σ-ACM accepts all Σ-dags. This latter question, called universality, essentially asks whether the described property is always satisfied.

The corresponding question for finite sequential automata has a positive answer which is a consequence of the decidability of the emptiness: If one wants to know whether a sequential automaton accepts all words, one constructs the complementary automaton and checks whether its language is empty. Thus, the crucial point for sequential automata is that they can effectively be complemented. But Example 2.1.6 shows that the set of acceptable Σ-dag-languages is not closed under complementation. Therefore, Theorem 3.3.5 does not imply that the universality of an Σ-ACM is decidable. On the contrary, we show that the universality is undecidable even for Σ-ACAs. This implies that the equivalence of two Σ-ACAs, the complementability and the determinisability of a Σ-ACA are undecidable, too. This result was announced in [Kus98] for Hasse-diagrams together with the sketch of a proof. This original proof idea used the undecidability of the Halting Problem. Differently, our proof here is based on the undecidability of the Tiling Problem. This change, as well as the formulation and proof of Lemmas 4.1.4 and 4.1.5 were obtained in collaboration with Paul Gastin. *Throughout this section, let* $\Sigma = \{a, b\}$ *if not stated otherwise.*

Let $\mathfrak{C}$ be a finite set of colors with white $\in \mathfrak{C}$. A mapping $\tau : \{\mathrm{W}, \mathrm{N}, \mathrm{E}, \mathrm{S}\} \to \mathfrak{C}$ is called a *tile*. Since the elements W, N etc. stand for the cardinal points, a tile can be visualized as follows:



Now let $\mathcal{T}$ be a set of tiles and $k, \ell \in \mathbb{N}^+$. A mapping $T : [k] \times [\ell] \to \mathcal{T}$ is a *tiling of the grid* $[k] \times [\ell]$ provided for any $(i, j) \in [k] \times [\ell]$ we have

1. $f(i,j)(\mathrm{W}) = \begin{cases} \text{white} & \text{if } i = 1 \\ f(i-1,j)(\mathrm{E}) & \text{otherwise} \end{cases}$

2. $f(i,j)(\mathrm{S}) = \begin{cases} \text{white} & \text{if } j = 1 \\ f(i,j-1)(\mathrm{N}) & \text{otherwise} \end{cases}$

Note that then $f(i,j)(\mathrm{E}) = f(i+1,j)(\mathrm{W})$ for $i < k$, and similarly $f(i,j)(\mathrm{N}) = f(i,j+1)(\mathrm{S})$ for $j < \ell$. An *infinite tiling* is a mapping $f : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathcal{T}$ such that for any $k \in \mathbb{N}^+$ the restriction of f to $[k] \times [k]$ is a tiling. It is known that for a set $\mathcal{T}$ of tiles the existence of an infinite tiling is undecidable [Ber66].

A set of grids is *unbounded* if, for any $k, \ell \in \mathbb{N}^+$, it contains a grid $[k'] \times [\ell']$ with $k \leq k'$ and $\ell \leq \ell'$.

Lemma 4.1.1 *Let $\mathcal{T}$ be a set of tiles for the finite set of colors $\mathfrak{C}$. Then $\mathcal{T}$ allows an infinite tiling iff the set of grids that allow a tiling is unbounded.*

Proof. Let $f : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathcal{T}$ be an infinite tiling. Then, for $k, \ell \in \mathbb{N}^+$, let $k' = \max(k, \ell)$. By definition, the restriction of f to $[k'] \times [k']$ is a tiling. Thus, the set of tilable grids is unbounded.

For the converse let T denote the set of all tilings of squares $[k] \times [k]$ for some $k \in \mathbb{N}^+$ ordered by inclusion. Then this is a tree. Any node of the tree has finitely many upper neighbors. Since the set of all tilable grids is unbounded, all squares can be tiled. Hence T is infinite. By König's Lemma, it has an infinite branch $(f_i)_{i \in \mathbb{N}^+}$. Then $f = \bigcup_{i \in \mathbb{N}^+} f_i$ is an infinite tiling. □

To encode the tiling problem into our setting of Σ-dags, we will consider the (k, ℓ)-grids $[k] \times [\ell]$ with $k, \ell \in \mathbb{N}^+$ and the edge relation

$$\begin{aligned} E' = \quad & \{((i,j),(i,j+1)) \mid 1 \leq i \leq k, 1 \leq j < \ell\} \cup \\ & \{((i,j),(i+1,j)) \mid 1 \leq i < k, 1 \leq j \leq \ell\}. \end{aligned}$$

Let $\leq$ be the reflexive and transitive closure of E'. Then the partial orders $([k] \times [\ell], \leq)$ contain antichains of size $\min(k, \ell)$. Hence they do not fit into our setting of Σ-dags where the size of antichains is restricted to n. Therefore, we define

$$((i,j),(i',j')) \in E \text{ iff } ((i,j),(i',j')) \in E' \text{ or } j+2=j', i=\ell, \text{ and } i'=1.$$

(see Figure 4.1). The Σ-dag $([k]\times[\ell], E, \lambda)$ is *the folding of the grid* $[k]\times[\ell]$ or a *folded grid*. Let $\preceq$ denote the transitive and reflexive closure of E. Then the partially ordered set $([k]\times[\ell], \preceq)$ contains antichains of size 2, only, and E is the covering relation of $\preceq$. Furthermore, the chains $\{(i,2j+1) \mid i \in [k], 2j+1 \in [\ell]\}$ and $\{(i,2j) \mid i \in [k], 2j \in [\ell]\}$ form a partition of the partial order $([k]\times[\ell], \preceq)$. We label the elements $(i,2j+1)$ of the first chain by a. Similarly, the elements $(i,2j)$ of the second chain are labeled by b. Thus, two elements get the same label iff their second components have the same parity. Note that in the folded grid all vertices except $(1,1)$ have a lower neighbor labeled by a, and that all vertices (i,j) with $j>1$ have a lower neighbor labeled by b. Hence for $1 \le i \le k$ and $1 \le j \le \ell$ it holds

$$\mathrm{R}(i,j) = \begin{cases} \emptyset & \text{for } i=j=1 \\ \{a\} & \text{for } 1<i\le k, j=1 \text{ or } (i,j)=(1,2) \\ \{a,b\} & \text{otherwise.} \end{cases}$$

Furthermore, we have

$$\partial_a(i,j) = \begin{cases} \text{undefined} & \text{for } i=j=1 \\ (i-1,j) & \text{for } 1<i\le k, j \text{ odd} \\ (k,j-2) & \text{for } i=1, 1<j\le \ell \text{ odd} \\ (i,j-1) & \text{for } j \text{ even, and} \end{cases}$$

$$\partial_b(i,j) = \begin{cases} \text{undefined} & \text{for } j=1 \text{ or } (i,j)=(1,2) \\ (i-1,j) & \text{for } 1<i\le k, j \text{ even} \\ (k,j-2) & \text{for } i=1, 1<j\le \ell \text{ even} \\ (i,j-1) & \text{for } j>2 \text{ odd.} \end{cases}$$

Let $\mathbb{G}$ comprise the set of all folded grids, i.e. we define $\mathbb{G} \subseteq \mathbb{D}$ by

$$\mathbb{G} = \{([k]\times[\ell], E, \lambda) \mid k,\ell \in \mathbb{N}^+\}.$$

Next, from a set of tiles $\mathcal{T}$, we construct a Σ-ACA $\mathcal{A}_{\mathcal{T}}$ that recognizes among all folded grids those that allow a tiling. This automaton guesses a tile for any vertex and checks that it is a tiling. This would be fairly easy if the Σ-ACA run on a grid, but it has to run on a folded grid. Hence, on the folded grid, it has to find out which edges belong to the original grid and which edges are new. In other words, it has to distinguish between nodes of the form $(1,j)$ and those of the form (i,j) with $1<i$.

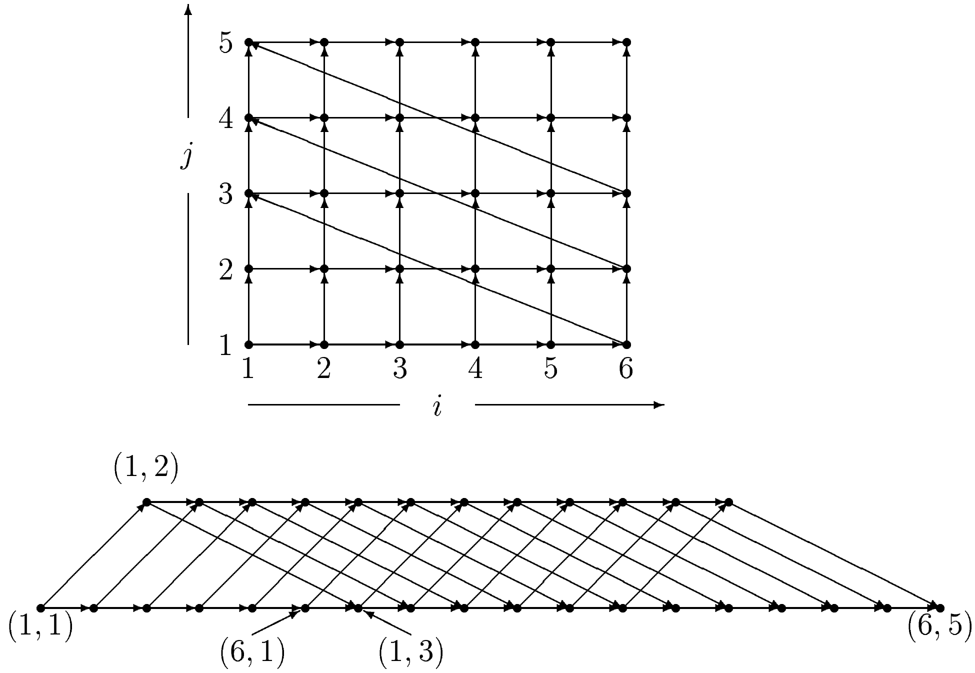


Figure 4.1: The folded grid $([6] \times [5], E)$

So let $\mathfrak{C}$ be a finite set of colors and $\mathcal{T}$ a set of tiles. Then the ACA $\mathcal{A}_\mathcal{T}$ is given by $Q_a = Q_b = \mathcal{T} \times \{0, 1\}$ and

$$\begin{aligned}
\delta_{a,\emptyset} &= \{g \in \mathcal{T} \mid g(\mathrm{W}) = g(\mathrm{S}) = \text{white}\} \times \{1\},\\
\delta_{a,\{a\}}((q_a, s_a)) &= \{g \in \mathcal{T} \mid g(\mathrm{W}) = q_a(\mathrm{E}), g(\mathrm{S}) = \text{white}\} \times \{0\},\\
\delta_{a,\Sigma}((q_c, s_c)_{c\in\Sigma}) &= \begin{cases} \{g \in \mathcal{T} \mid g(\mathrm{W}) = \text{white}, g(\mathrm{S}) = q_b(\mathrm{N})\} \times \{1\} & \text{if } s_b = 1 \\ \{g \in \mathcal{T} \mid g(\mathrm{W}) = q_a(\mathrm{E}), g(\mathrm{S}) = q_b(\mathrm{N})\} \times \{0\} & \text{if } s_b = 0, \end{cases}\\
\delta_{b,\Sigma}((q_c, s_c)_{c\in\Sigma}) &= \begin{cases} \{g \in \mathcal{T} \mid g(\mathrm{W}) = \text{white}, g(\mathrm{S}) = q_a(\mathrm{N})\} \times \{1\} & \text{if } s_a = 1 \\ \{g \in \mathcal{T} \mid g(\mathrm{W}) = q_b(\mathrm{E}), g(\mathrm{S}) = q_a(\mathrm{N})\} \times \{0\} & \text{if } s_a = 0. \end{cases}
\end{aligned}$$

All tuples of local states are accepting. Now let $t = (V, E, \lambda)$ be a folded grid with $V = [k] \times [\ell]$ and let f be a tiling of this grid. We define a mapping $r : [k] \times [\ell] \to \mathcal{T}$ by

$$r((i,j)) = \begin{cases} (f(i,j), 1) & \text{if } i = 1 \\ (f(i,j), 0) & \text{if } i > 1 \end{cases}$$

and show that it is a successful run of $\mathcal{A}_\mathcal{T}$: Since f is a tiling of $[k] \times [\ell]$, we get $f(1,1)(\mathrm{W}) = f(1,1)(\mathrm{S}) = \text{white}$. Hence $r(1,1) = (f(1,1), 1) \in \delta_{a,\emptyset} = \delta_{\lambda(1,1),\mathrm{R}(1,1)}$. Now let $1 < i \le \ell$. Since f is a tiling, we have $f(i,1)(\mathrm{W}) = f(i-1,1)(\mathrm{E})$

and $f(i,1)(\mathrm{S}) = \text{white}$. Hence $r(i,1) = (f(i,1),0) \in \delta_{a,\{a\}}((f(i-1,1),s_a))$ for any $s_a \in \{0,1\}$. Note that $\lambda(i,1) = a$ and $\mathrm{R}(i,1) = \{a\}$. Furthermore, $\partial_a((i,1)) = (i-1,1)$. Hence we get $r(i,1) \in \delta_{\lambda(i,1),\mathrm{R}(i,1)}(r\partial_a((i,1)))$, i.e. the run condition of $\mathcal{A}_{\mathcal{T}}$ is satisfied at all nodes of the form $(i,1)$ with $i \in [k]$.

Next consider a vertex $(1,j)$ with $1 < j \leq k$ odd. Then $r(1,j-1)$ equals $(f(1,j-1),1)$. Since f is a tiling, we obtain $f(1,j)(\mathrm{W}) = \text{white}$ and $f(1,j)(\mathrm{S}) = f(1,j-1)(\mathrm{N})$. Hence

$$r(1,j) = (f(1,j),1) \in \delta_{a,\Sigma}(r(k,j-2), r(1,j-1)).$$

Since $j > 2$ is odd, $\lambda(1,j) = a$ and $\mathrm{R}(1,j) = \Sigma$. From $3 \leq j$ we get $(k,j-2) = \partial_a((1,j))$ and $(1,j-1) = \partial_b((1,j))$. Thus, we showed

$$r(1,j) \in \delta_{\lambda(1,j),\mathrm{R}(1,j)}(r\partial_a((1,j)), r\partial_b((1,j))).$$

For j even we can argue similarly. Hence we showed that the run condition of $\mathcal{A}_{\mathcal{T}}$ is satisfied at all nodes of the form $(i,1)$ or $(1,j)$ with $i \in [k]$ and $j \in [\ell]$.

It remains to consider a vertex (i,j) with $1 < i \leq k$ and $1 < j \leq \ell$. Assume j to be even. Since $i > 1$, $r(i,j-1) = (f(i,j-1),0)$. Since f is a tiling, we have $f(i,j)(\mathrm{W}) = f(i-1,j)(\mathrm{E})$ and $f(i,j)(\mathrm{S}) = f(i,j-1)(\mathrm{N})$. Hence $r(i,j) = (f(i,j),0) \in \delta_{b,\Sigma}(r(i,j-1), r(i-1,j))$. Since j is even, $\lambda(i,j) = b$. Note that $\mathrm{R}(i,j) = \Sigma$, $\partial_a((i,j)) = (i,j-1)$ and $\partial_b((i,j)) = (i-1,j)$. Hence we have $r(i,j) \in \delta_{\lambda(i,j),\mathrm{R}(i,j)}(r\partial_a((i,j)), r\partial_b((i,j)))$. Again, for j odd we can argue similarly. Thus the mapping r is a run of the ACA $\mathcal{A}_{\mathcal{T}}$ on t. Since any tuple is accepting, $t \in L(\mathcal{A}_{\mathcal{T}})$. Thus we showed that $\mathcal{A}_{\mathcal{T}}$ accepts all foldings of grids that allow a tiling.

Conversely, let r be a successful run of $\mathcal{A}_{\mathcal{T}}$ on the folded grid $t = (V, E, \lambda)$ with $V = [k] \times [\ell]$. We show that $f := \pi_1 \circ r$ is a tiling: First observe that $\pi_2 \circ r(i,j) = 1$ iff (i,j) is minimal in $(V, \preceq)$ or (i,j) has a lower neighbor $x \in V$ with $\lambda(x) \neq \lambda(i,j)$ and $\pi_2 \circ r(x) = 1$. Since $(i,j-1)$ is the only possible lower neighbor with a different label, $\pi_2 \circ r(i,j) = 1$ iff $i = j = 1$ or $\pi_2 \circ r(i,j-1) = 1$. Hence by induction $\pi_2 \circ r(i,j) = 1$ iff $i = 1$.

Since r is a run and $\lambda(1,1) = a$, we obtain $r(1,1) \in \delta_{\lambda(1,1),\emptyset}$. Hence $f(1,1)(\mathrm{W})$ and $f(1,1)(\mathrm{S})$ both equal white, i.e. f satisfies the conditions for a tiling at the point $(1,1)$.

Next let $1 < i \leq k$. Then $\mathrm{R}(i,1) = \{a\}$, $\lambda(i,1) = a$ and $\partial_a((i,1)) = (i-1,1)$. Since r is a run, this implies $r(i,j) \in \delta_{a,\{a\}}(r(i-1,1))$. The definition of $\delta_{a,\{a\}}$ implies $f(i,1)(\mathrm{W}) = f(i-1,1)(\mathrm{E})$ and $f(i,1)(\mathrm{S}) = \text{white}$ since $\pi_1 \circ r(i-1,1) = f(i-1,1)$. Hence $f \restriction ([k] \times [1])$ is a tiling.

Now let $1 < j \leq \ell$ be odd. Then $\mathrm{R}(1,j) = \Sigma$, $\lambda(1,j) = a$, $\partial_a((1,j)) = (k,j-2)$ and $\partial_b((1,j)) = (1,j-1)$. Since r is a run, this implies

$$r(1,j) \in \delta_{a,\Sigma}(r(k,j-2), r(1,j-1)).$$

Note that $\pi_2 \circ r(1, j-1) = 1$. Hence by the definition of $\delta_{a,\Sigma}$, $f(1,j)(\mathrm{W}) = \text{white}$ and $f(1,j)(\mathrm{S}) = f(1,j-1)(\mathrm{N})$. Since we can argue similarly for j even, the restriction $f \restriction ([1] \times [\ell])$ of f is a tiling.

It remains to consider the case $1 < i \leq k$ and $1 < j \leq \ell$. Then $\mathrm{R}(i,j) = \Sigma$. Now let j be even. Then $\lambda(i,j) = b$, $\partial_a((i,j)) = (i, j-1)$ and $\partial_b((i,j)) = (i-1, j)$. Since r is a run, $r(i,j) \in \delta_{b,\Sigma}(r(i,j-1), r(i-1,j))$. Since $i > 0$, we have $\pi_2 \circ r(i, j-1) = 0$. Thus the definition of $\delta_{b,\Sigma}$ yields $f(i,j)(\mathrm{W}) = f(i-1,j)(\mathrm{E})$ and $f(i,j)(\mathrm{S}) = f(i,j-1)(\mathrm{N})$. Again, for j odd we can argue similarly. Thus, f is indeed a tiling of the grid $[k] \times [\ell]$, i.e. we proved

Lemma 4.1.2 *Let t be the folding of the grid $[k] \times [\ell]$. Then $t \in L(\mathcal{A}_{\mathcal{T}})$ iff $[k] \times [\ell]$ admits a tiling. In particular, $L(\mathcal{A}_{\mathcal{T}}) \cap \mathbb{G}$ is the set of all foldings of tilable grids.*

Note that $\mathcal{A}_{\mathcal{T}}$ accepts the foldings of an unbounded set of grids iff it accepts all folded grids. Lemma 4.1.1 and 4.1.2 imply that $\mathcal{A}_{\mathcal{T}}$ accepts an unbounded set of grids iff $\mathcal{T}$ admits an infinite tiling. Since the existence of an infinite tiling is undecidable, it is undecidable whether a given Σ-ACA $\mathcal{A}$ accepts the foldings of an unbounded set of grids and therefore whether $\mathbb{G} \subseteq L(\mathcal{A})$. Since $\mathbb{G}$ is not recognizable (cf. Lemma 4.1.9 below), this result cannot be used immediately to show the undecidability of the equivalence of ACAs. Nevertheless, it is a milestone in our proof that continues by showing that $\mathbb{D} \setminus \mathbb{G}$ is recognizable. This will imply that for a tiling systems $\mathcal{T}$ the set of all Σ-dags that are

a) no folded grid, or

b) a folded grid that can be tiled

is recognizable. But this set equals $\mathbb{D}$ iff the tiling system $\mathcal{T}$ allows an infinite tiling, and the latter is undecidable. Thus, indeed, it remains to show that $\mathbb{D} \setminus \mathbb{G}$ is recognizable.

Recall that $\mathrm{Ha} \subseteq \mathbb{D}$ is the set of Hasse-diagrams in $\mathbb{D}$. It is easily seen that $(V, E, \lambda) \in \mathbb{D}$ belongs to Ha iff it satisfies

$$(x,z),(y,z) \in E \Longrightarrow (x,y) \notin E^+$$

for all $x, y, z \in V$. Then $\mathbb{G} \subseteq \mathrm{Ha} \subseteq \mathbb{D}$ implies $\mathbb{D} \setminus \mathbb{G} = \mathbb{D} \setminus \mathrm{Ha} \cup \mathrm{Ha} \setminus \mathbb{G}$.

By Example 2.1.3, the set of Hasse-diagrams can be accepted by a Σ-ACM. Next, we prove that the *complement* of this set can be accepted using only finitely many states, i.e. by a Σ-ACA:

Lemma 4.1.3 *There exists a Σ-ACA $\mathcal{A}_{\mathrm{Ha}^{co}}$ with $L(\mathcal{A}_{\mathrm{Ha}^{co}}) = \mathbb{D} \setminus \mathrm{Ha}$.*

Proof. We present an automaton $\mathcal{A}_a$ that recognizes all Σ-dags (V, E, λ) satisfying

there are an a-labeled vertex x and vertices y and z with $(x,z),(y,z) \in E$ and $(x,y) \in E^+$. $(\star)$

Let $\mathcal{A}_b$ be the analogous automaton that accepts all Σ-dags satisfying the above condition where x is supposed to carry the label b. Then $\mathbb{D}\setminus\mathrm{Ha} = L(\mathcal{A}_a) \cup L(\mathcal{A}_b)$ is recognizable.

To construct $\mathcal{A}_a$, let $Q_a = Q_b = \{0,1,2,3\}$. Then, the transition functions are defined as follows:

$$\delta_{a,J}((q_j)_{j\in J}) = \begin{cases} \{0,1\} & \text{if } \{q_j \mid j \in J\} \subseteq \{0\} \\ \{3\} & \text{if } 3 \in \{q_j \mid j \in J\} \\ \{2,3\} & \text{if } \{q_j \mid j \in J\} = \{1,2\} \\ \{2\} & \text{otherwise, and} \end{cases}$$

$$\delta_{b,J}((q_j)_{j\in J}) = \begin{cases} \{0\} & \text{if } \{q_j \mid j \in J\} \subseteq \{0\} \\ \{3\} & \text{if } 3 \in \{q_j \mid j \in J\} \\ \{2,3\} & \text{if } \{q_j \mid j \in J\} = \{1,2\} \\ \{2\} & \text{otherwise.} \end{cases}$$

A tuple of states is accepting, i.e. belongs to F, if it contains the local state 3.

Let $t = (V, E, \lambda) \in \mathbb{D}$ satisfy $(\star)$. Then there are x, y, z with $\lambda(x) = a$, $(x,z), (y,z) \in E$ and $(x,y) \in E^+$. Define a mapping (cf. Figure 4.2) $r : V \to Q$ by

$$r(v) := \begin{cases} 0 & \text{if } x \not\leq v \\ 1 & \text{if } v = x \\ 3 & \text{if } z \leq v \\ 2 & \text{otherwise.} \end{cases}$$

In Figure 4.2, this mapping is depicted. There, solid vectors correspond to edges from E, the dotted vector connecting x and y denotes that $(x,y) \in E^+$. Furthermore, the dashed lines indicate the borders between e.g. $r^{-1}(0)$ and $r^{-1}(2)$, the values taken by r in an area is written there. Note that the small triangle around x depicts $r^{-1}(1)$ and contains x only.

We have to show $r(v) \in \delta_{\lambda(v),\mathrm{R}(v)}((r\partial_b(v))_{b\in\mathrm{R}(v)})$ $(\star\star)$ for any $v \in V$: Note that $r^{-1}(3)$ is a principal filter. Each nonminimal element of this filter reads a state 3, i.e. these elements satisfy $(\star\star)$. Since $(x,z), (y,z) \in E$ and x and y are different, they carry different labels. Hence $\lambda(x) = a$ implies $\lambda(y) = b$. Thus we have $\mathrm{R}(z) = \Sigma$, $r\partial_a(z) = 1$ and $r\partial_b(z) = 2$. Hence $(\star\star)$ holds for the minimal element z of this principal filter, too. The set $r^{-1}(2) = \{v \in V \mid x < v, z \not\leq v\}$ is convex. Note that $2 \in \delta_{c,J}((q_d)_{d\in J})$ iff $3 \notin \{q_d \mid d \in J\} \not\subseteq \{0\}$. Now let $v \in r^{-1}(2)$. Since $z \not\leq v$, $3 \notin \{r\partial_c(v) \mid c \in \mathrm{R}(v)\}$. If v is nonminimal in $r^{-1}(2)$, it satisfies $(\star\star)$ since it reads the state 2. The minimal elements read the state at the vertex x which equals 1. Hence they satisfy $(\star\star)$, too. Note that $\{\partial_c(x) \mid c \in \mathrm{R}(x)\} \subseteq r^{-1}(0)$. Hence x satisfies $(\star\star)$. Since, finally, $r^{-1}(0)$ is an order ideal, $(\star\star)$ holds for its elements, too. Thus, r is a successful run of $\mathcal{A}_a$.

Conversely, let r be a successful run of $\mathcal{A}_a$ on a Σ-dag $t = (V, E, \lambda)$. For simplicity, let $\leq := E^\star$ denote the partial order induced by the edge relation E.

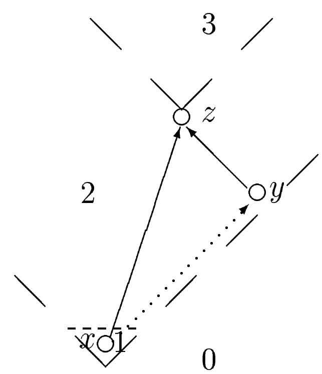


Figure 4.2: cf. Proof of Lemma 4.1.3

Then $r^{-1}(3)$ is a filter, i.e. an upward closed subset of V with respect to $\leq$. Since the run is successful, this filter is not empty. Note that an element of $r^{-1}(3)$ is minimal in this filter iff it reads a state 1 and a state 2. Since the filter in question is not empty, it contains a minimal element z and there are elements $x, y \in V$ with $(x, z), (y, z) \in E$, $r(x) = 1$ and $r(y) = 2$. Whenever a vertex v carries the state 2, it reads the state 1 or the state 2. Hence, by induction, we find $x' \in V$ with $(x', y) \in E^+$ and $r(x') = 1$. Since $r(x) = r(x') = 1$, they both carry the label a implying that they are comparable. Furthermore, $\{r\partial_c(x) \mid c \in \mathrm{R}(x)\} \subseteq \{0\}$ and $\{r\partial_c(x') \mid c \in \mathrm{R}(x')\} \subseteq \{0\}$. Since, as is easy to see, $r^{-1}(0)$ is an order ideal (i.e. downword closed), this implies $x = x'$. Hence $(x, y) \in E^+$, i.e. all Σ-dags accepted by $\mathcal{A}_a$ satisfy the condition $(\star)$. $\square$

Before showing that $\mathrm{Ha} \setminus \mathbb{G}$ is recognizable relative to Ha, we need an internal characterization of those Hasse-diagrams that are folded grids. This characterization is based on the notion of an *alternating covering chain*: Let a Hasse-diagram $t = (V, -\!\!<, \lambda) \in \mathrm{Ha}$ and a set $C \subseteq V$ be given. The set C is called *alternating covering chain* if it is a chain (with respect to $\leq := -\!\!<^\star$) such that

1. for all $y \in C$ with $y \neq \min(C)$, there exists $x \in C$ with $x -\!\!< y$ and $\lambda(x) \neq \lambda(y)$, and

2. for all $y \in C$ with $y \neq \max(C)$, there exists $z \in C$ with $y -\!\!< z$ and $\lambda(y) \neq \lambda(z)$.

Since we consider only Hasse-diagrams of width 2, it is easy to see that for any $x \in V$ there exists a unique maximal alternating covering chain C with $x \in C$. This chain is denoted by $C(x)$.

Lemma 4.1.4 *Let $t = (V, \multimap, \lambda) \in \mathrm{Ha}$ be a Hasse-diagram. Then $t \in \mathbb{G}$ if and only if*

(1) for any $x \in V$, the element $\min C(x)$ does not dominate any b-labeled vertex, and

(2) for any $x, y \in V$ with $x \multimap y$ such that y does not dominate any b-labeled element, we have (A) $\forall x' \in C(x) \exists y' \in C(y) : x' \multimap y'$
(B) $\forall y' \in C(y) \exists x' \in C(x) : x' \multimap y'$.

Proof. First, let $k, \ell \in \mathbb{N}^+$ and define $K_i = \{i\} \times [\ell]$ for $1 \leq i \leq k$. Then, in the folded grid $([k] \times [\ell], E)$, K_i is a chain. Since $((i,j),(i,j+1)) \in E$ and $\lambda(i,j) = a$ iff j is odd, it is even an alternating covering chain. We show that it is maximal: Let $x \multimap \min(K_i) = (i,1)$. Then $x = (i-1,1)$ and therefore $\lambda(x) = a = \lambda(i,1)$. Hence K_i cannot be extended downwards. Similarly, let $y \in [k] \times [\ell]$ with $(i,\ell) = \max K_i \multimap y$. Then $y = (i+1,\ell)$ and therefore carries the same label as (i,ℓ). Hence K_i is indeed a maximal alternating covering chain. Hence, for $(i,j) \in [k] \times [\ell]$, $C(i,j) = K_i$. Now it is routine to check properties (1) and (2) (cf. Figure 4.1).

Conversely, suppose $t = (V, \multimap, \lambda)$ satisfies the conditions (1) and (2) and let $\leq$ denote the transitive and reflexive closure of $\multimap$. By (1), $\lambda(\min C(x)) = a$ for any $x \in V$. Now let $\{a_1, a_2, \ldots, a_k\} = \{\min C(x) \mid x \in V\}$. Since each a_i is labeled by a, this set forms a chain. So let $a_1 < a_2 \cdots < a_k$. Since (again by (1)) none of the elements a_i dominates a b-labeled vertex, we have even $a_1 \multimap a_2 \ldots \multimap a_k$. For simplicity, let $C_i := C(a_i)$. The tuple $(C_i)_{i \in [k]}$ is a partition of V. We denote the jth element of the ith alternating covering chain C_i by x_i^j, i.e. $C_i = \{x_i^1, x_i^2, x_i^3 \ldots, x_i^{\ell_i}\}$ with $a_i = x_i^1 \multimap x_i^2 \multimap x_i^3 \ldots \multimap x_i^{\ell_i}$. Note that $\lambda(x_i^j) = a$ iff j is odd and that ℓ_i is the size of the chain C_i.

Claim 1 for any $1 \leq i < k$ and any $1 \leq j \leq \ell_i$ we have $j \leq \ell_{i+1}$ and $x_i^j \multimap x_{i+1}^j$.

This is shown inductively on j. Clearly, $1 \leq \ell_{i+1}$ since $a_{i+1} = x_{i+1}^1$. We already remarked that $x_i^1 \multimap x_{i+1}^1$. Now suppose $1 < j \leq \ell_i$ and $x_i^{j-1} \multimap x_{i+1}^{j-1}$. We can apply (2A) since $x_i^1 \multimap x_{i+1}^1$ and $x_i^j \in C_i$. Hence there exists $y' \in C_{i+1}$ with $x_i^j \multimap y'$. Since C_{i+1} is a chain containing x_{i+1}^{j-1} and y', these two elements are comparable. If $y' \leq x_{i+1}^{j-1}$, we had $x_i^{j-1} \multimap x_i^j \multimap y' \leq x_{i+1}^{j-1}$, contradicting $x_i^{j-1} \multimap x_{i+1}^{j-1}$. Hence $x_{i+1}^{j-1} < y'$. Since they both belong to the alternating covering chain C_{i+1}, there exists $y'' \in C_{i+1}$ with $x_{i+1}^{j-1} \multimap y'' \leq y'$. From $x_i^1 \multimap x_{i+1}^1$ and $y'' \in C_{i+1}$, we obtain by (2B) the existence of $x'' \in C_i$ with $x'' \multimap y''$. Now the elements x_i^{j-1} and x'' are comparable. If $x'' \leq x_i^{j-1}$, we had $x'' \leq x_i^{j-1} \multimap x_{i+1}^{j-1} \multimap y''$, contradicting $x'' \multimap y''$. Hence $x_i^{j-1} < x''$. Since $x'', x_i^j \in C_i$, they are comparable. Now $x_i^{j-1} \multimap x_i^j$ implies $x_i^j \leq x''$. Hence we have $x_i^j \leq x'' \multimap y'' \leq y'$ and $x_i^j \multimap y'$. This implies $y' = y''$. Recall that $x_{i+1}^{j-1} \multimap y'' \in C_{i+1}$. Hence we showed $y' = x_{i+1}^j$, i.e. $j \leq \ell_{i+1}$ and $x_i^j \multimap x_{i+1}^j$ as claimed.

Claim 2 For any $1 \leq i < k$ and any $1 \leq j \leq \ell_{i+1}$, we have $j \leq \ell_i$ and $x_i^j \longrightarrow\!\!< x_{i+1}^j$.

Again, this is shown by induction on j. Clearly, $1 \leq \ell_i$ since $a_i = x_i^1$. Now, $x_i^1 \longrightarrow\!\!< x_{i+1}^1$ follows from Claim 1. Now suppose $1 < j \leq \ell_i$ such that $x_i^{j'} \longrightarrow\!\!< x_{i+1}^{j'}$ for any $j' < j$. Then we can apply (2B) since $x_i^1 \longrightarrow\!\!< x_{i+1}^1$ and $x_{i+1}^j \in C_i$. Hence there exists $x' \in C_i$ with $x' \longrightarrow\!\!< x_{i+1}^j$. Since $x_i^{j'} \longrightarrow\!\!< x_{i+1}^{j'}$ for $j' < j$, we have $x' \neq x_i^{j'}$ for $j' < j$. Hence the chain C_i contains at least j elements, i.e. $j \leq \ell_i$. Now $x_i^j \longrightarrow\!\!< x_{i+1}^j$ follows from Claim 1.

Note that Claim 1 in particular implies $\ell_1 \leq \ell_2 \cdots \leq \ell_k$. Similarly, by Claim 2, $\ell_1 \geq \ell_2 \cdots \geq \ell_k$, i.e. $\ell_1 = \ell_2 \cdots = \ell_k =: \ell$. Hence $g : [k] \times [\ell] \to V : (i,j) \mapsto x_i^j$ is a bijection.

Claim 3 For $1 \leq i \leq i' \leq k$ and $1 \leq j \leq \ell$, $x_{i'}^j$ is the least element of $C_{i'}$ dominating x_i^j, i.e. $x_{i'}^j = \min\{x \in C_{i'} \mid x_i^j \leq x\}$.

This is trivial for $i = i'$. For $i + 1 = i'$ it is clear by Claim 1. By induction, suppose we showed that $x_{i'-1}^j$ is the least element of $C_{i'-1}$ that dominates x_i^j. Let $x_{i'}^{j'}$ be the least element of $C_{i'}$ dominating x_i^j. Since $|\Sigma| = 2$, this element $x_{i'}^{j'}$ has at most two lower neighbors, namely $x_{i'}^{j'-1}$ (if $j' > 1$) since it precedes $x_{i'}^{j'}$ in the alternating covering chain $C_{i'}$, and $x_{i'-1}^{j'}$ by Claim 1. Since x_i^j is not dominated by $x_{i'}^{j'-1} \in C_{i'}$, we therefore have $x_{i'-1}^{j'} \geq x_i^j$. Hence, by the induction hypothesis, $j' \geq j$. Since $x_i^j \leq x_{i'-1}^j \longrightarrow\!\!< x_{i'}^j$, we therefore showed that $x_{i'}^j$ is the least element in $C_{i'}$ dominating x_i^j.

Now we show that the bijection $g : ([k] \times [\ell], E) \to (V, \longrightarrow\!\!<)$ is order preserving: Let $(i,j), (i',j') \in [k] \times [\ell]$ with $((i,j),(i',j')) \in E$. Then we have

a) $i = i'$ and $j + 1 = j'$, or
b) $i + 1 = i'$ and $j = j'$, or
c) $i = k$, $i' = 1$ and $j + 2 = j'$.

In the first case, we get immediately $x_i^j \longrightarrow\!\!< x_{i'}^{j'}$ since they are consecutive elements of the alternating covering chain C_i. In the second case, Claim 1 implies $x_i^j \longrightarrow\!\!< x_{i'}^{j'}$. In the third case, we get $g(i,j) = x_k^j$ and $g(i',j') = x_1^{j+2}$. Since j and $j + 2$ have the same parity, $\lambda(x_k^j) = \lambda(x_1^{j+2})$ and therefore x_k^j and x_1^{j+2} are comparable. If $x_1^{j+2} \leq x_k^j$, by Claim 3 we have $x_k^j \geq x_k^{j+2}$ which is properly larger than x_k^j, a contradiction. Hence $x_k^j < x_1^{j+2}$, i.e. $g(i,j) < g(i',j')$. Thus we showed that g is order preserving.

Next we show that $f = g^{-1} : (V, \longrightarrow\!\!<) \to ([k] \times [\ell], E)$ is order preserving. So let $x_i^j, x_{i'}^{j'} \in V$ with $x_i^j \longrightarrow\!\!< x_{i'}^{j'}$. If x_i^j and $x_{i'}^{j'}$ carry the same label, j and j' have the same parity. Hence $\lambda(i,j) = \lambda(i',j')$. This ensures that (i,j) and (i',j') are comparable. If $(i',j') \preceq (i,j)$, we get $x_{i'}^{j'} = g(i',j') \leq g(i,j) = x_i^j$ since g is order preserving. But this contradicts the assumption $x_i^j \longrightarrow\!\!< x_{i'}^{j'}$. Hence $(i,j) < (i',j')$, i.e. $f(x_i^j) < f(x_{i'}^{j'})$. Now assume that x_i^j and $x_{i'}^{j'}$ carry different labels. Since $x_i^j \longrightarrow\!\!< x_{i'}^{j'}$ this implies that they belong to the same maximal alternating covering chain, i.e. $i = i'$ and $j + 1 = j'$. But then $f(x_i^j) = (i,j)E(i,j+1) = f(x_{i'}^{j'})$. $\square$

This characterization of the Hasse-diagrams of folded grids enables us to show that $\mathbb{D} \setminus \mathbb{G}$ is recognizable by a Σ-ACA:

Lemma 4.1.5 *There exists a Σ-ACA $\mathcal{A}$ such that $L(\mathcal{A}) = \mathbb{D} \setminus \mathbb{G}$.*

Proof. By Lemma 4.1.3, there exists a Σ-ACA $\mathcal{A}_{\mathrm{Ha}^{co}}$ with $L(\mathcal{A}_{\mathrm{Ha}^{co}}) = \mathbb{D} \setminus \mathrm{Ha}$. From $\mathbb{G} \subseteq \mathrm{Ha} \subseteq \mathbb{D}$, we get $\mathbb{D} \setminus \mathbb{G} = \mathbb{D} \setminus \mathrm{Ha} \cup \mathrm{Ha} \setminus \mathbb{G}$. Hence it suffices to construct a Σ-ACA $\mathcal{A}$ with $L(\mathcal{A}) \cap \mathrm{Ha} = \mathrm{Ha} \setminus \mathbb{G}$.

As a prerequisite, we give an ACA $\mathcal{A}^1$ that marks all vertices which dominate a b-labeled vertex: Let $Q^1_a = Q^1_b = \{0,1\}$ where 0 stands for "does not dominate any b-labeled vertex". The transition functions are defined by $\delta^1_{b,J}((q_c)_{c\in J}) = \{1\}$ and

$$\delta^1_{a,J}((q_c)_{c\in J}) = \begin{cases} \{1\} & \text{if } 1 \in \{q_c \mid c \in J\} \\ \{0\} & \text{otherwise} \end{cases}$$

for any $J \subseteq \Sigma$ and $q_c \in Q^1_c$. Then, obviously, for any run r of $\mathcal{A}^1$ on a Hasse-diagram $t = (V, -\!\!<, \lambda)$, we have $r(x) = 0$ iff $b \notin \{\lambda(y) \mid y \leq x\}$ as claimed.

Next we prove that the set of Hasse-diagrams violating Lemma 4.1.4(1) can be accepted by a Σ-ACA relative to Ha: Note that Lemma 4.1.4(1) is violated iff there exists an a-labeled vertex x that dominates, but does not cover any b-labeled vertex. To find such a vertex, we enrich the automaton $\mathcal{A}^1$ by a second component that propagates the information whether a transition of the form $\delta_{a,\{a\}}(1)$ has been applied. If the run of this enriched automaton uses such a transition, it accepts, otherwise, it rejects. Note that the application of the transition $\delta_{a,\{a\}}(1)$ at a vertex x denotes that x is a-labeled, does not cover any b-labeled vertex, and dominates such a vertex according to the definition of $\mathcal{A}^1$. Hence the enriched automaton $\mathcal{A}_{\neg(1)}$ accepts precisely those Σ-pomsets that violate Lemma 4.1.4(1).

It remains to prove that the negation of statement (2) of Lemma 4.1.4 can be recognized. First, we show how to guess the element x and to mark the chain $C(x)$: Let $Q' = \{0,1,2\}$ where 2 stands for "belongs to $C(x)$", 1 for "does not belong to $C(x)$, but dominates an element of $C(x)$", and 0 for "does not dominate any element from $C(x)$. The transition functions of the automaton $\mathcal{A}'$ are given by

$$\delta'_{a,J}((q_c)_{c\in J}) = \begin{cases} \{2\} & \text{if } b \in J, q_b = 2 \\ \{1\} & \text{if } a \in J, q_a > 0 \text{ or } b \in J, q_b = 1 \\ \{0,2\} & \text{otherwise} \end{cases}$$

$$\delta'_{b,J}((q_c)_{c\in J}) = \begin{cases} \{2\} & \text{if } a \in J, q_a = 2 \\ \{1\} & \text{if } b \in J, q_b > 0 \text{ or } a \in J, q_a = 1 \\ \{0\} & \text{otherwise.} \end{cases}$$

Let r be a run of this automaton on the Hasse-diagram $t = (V, -\!\!<, \lambda)$. If $x \in V$ with $r(x) > 0$ then, either x covers some y with $r(y) > 0$, or $\lambda(x) = a$. Hence the set of all vertices x with $r(x) > 0$ (if not empty) is a principal filter (with respect to the partial order $\leq$ induced by $-\!\!<$) whose minimal element is labeled by a. In this principal filter, $r(x) = 2$ holds iff x covers some y with different label and $r(y) = 2$, or x is the minimal element of the principal filter. Hence, the set of all $x \in V$ with $r(x) = 2$ forms an alternating covering chain whose least element is labeled by a. Using the automaton $\mathcal{A}^1$, it is easily possible to ensure that this minimal element does not dominate any b-labeled vertex. Thus, we can construct a Σ-ACA $\mathcal{A}^2 = ((Q_a^2, Q_b^2), (\delta_{a,J}^2, \delta_{b,J}^2)_{J\subseteq\Sigma}, F^2)$ and subsets $S^x, S^y \subseteq Q_a^2 \cup Q_b^2$ such that for any successful run r on a Hasse-diagram $t = (V, -\!\!<, \lambda)$, we have

(a) $r^{-1}(S^x)$ and $r^{-1}(S^x)$ form nonempty alternating covering chains with minimal elements x and y,

(b) x and y do not dominate any b-labeled vertex, and

(c) $x -\!\!< y$.

Note that $t = (V, -\!\!<, \lambda)$ violates Lemma 4.1.4(2B) iff there exists a successful run r of the ACA $\mathcal{A}^2$ on t and an element $y' \in V$ with $r(y') \in S^y$ that does not cover any $x' \in V$ with $r(x') \in S^x$. Since this can easily be checked, we are therefore able to construct a Σ-ACA $\mathcal{A}_{\neg(2B)}$ such that $L(\mathcal{A}_{\neg(2B)}) \cap \mathrm{Ha}$ is the set of all Hasse-diagrams t that violate Lemma 4.1.4(2B).

To check the negation of Lemma 4.1.4(2A), we again use the automaton $\mathcal{A}^2$ that marks nondeterministically two alternating covering chains $C(x)$ and $C(y)$. This automaton will be enriched by the ability to mark some vertices from $C(x)$ and check that they are not covered by any element from $C(y)$. More formally, let $Q^3 = Q^2 \times \{0,1\}$. For $z \in \{a,b\}$, the transition function is given by

$$\delta_{z,J}^3((q_c, s_c)_{c\in J}) = \begin{cases} (\delta_{z,J}^2((q_c)_{c\in J}) \setminus S^y) \times \{0,1\} & \text{if } \exists c \in J : (q_c \in S^x \wedge s_c = 1) \\ \delta_{z,J}^2((q_c)_{c\in J}) \times \{0,1\} & \text{otherwise.} \end{cases}$$

Let r be a successful run of $\mathcal{A}^3$ on a Hasse-diagram $t = (V, -\!\!<, \lambda)$. Then $C(x) = r^{-1}(S^x \times \{0,1\})$ and $C(y) = r^{-1}(S^y \times \{0,1\})$ are alternating covering chains. Now suppose there is some $x' \in V$ with $r(x) \in S^x \times \{1\}$. Then, according to the definition of the transition function $\delta_{z,J}$, there is no $y' \in C(y)$ with $x' -\!\!< y'$ (since otherwise $r(y') \notin S^y \times \{0,1\}$, a contradiction). Hence, in this case, (2A) does not hold. Since the existence of a vertex x with $r(x) \in S^x \times \{1\}$ is easily checked, we can construct a Σ-ACA $\mathcal{A}_{\neg(2A)}$ such that $L(\mathcal{A}_{\neg(2A)}) \cap \mathrm{Ha}$ is the set of all Hasse-diagrams t that violate condition Lemma 4.1.4(2A).

Combining the automata $\mathcal{A}_{\neg(1)}$, $\mathcal{A}_{\neg(2B)}$ and $\mathcal{A}_{\neg(2A)}$, we get a Σ-ACA $\mathcal{A}'$ such that $L(\mathcal{A}') \cap \mathrm{Ha}$ is the set of all Hasse-diagrams t that violate condition (1) or (2). Using Lemma 4.1.3 and Lemma 4.1.4, one gets a Σ-ACA $\mathcal{A}$ with $L(\mathcal{A}) = \mathbb{D} \setminus \mathbb{G}$. $\square$

Putting the results obtained so far together, we find a Σ-ACA that accepts all folded grids that allow a tiling as well as all Σ-dags that are no folded grid:

Lemma 4.1.6 *Let $\mathcal{T}$ be a set of tiles. Then there exists a Σ-ACA $\mathcal{A}(\mathcal{T})$ such that $L(\mathcal{A}(\mathcal{T}))$ is the set of all Σ-dags that are no folded grid or a folding of a grid that allows a tiling by $\mathcal{T}$.*

Proof. By Lemma 4.1.5, there exists a Σ-ACA $\mathcal{A}'$ with $L(\mathcal{A}') = \mathbb{D} \setminus \mathbb{G}$. By Lemma 4.1.2, $L(\mathcal{A}_\mathcal{T}) \cap \mathbb{G}$ is the set of all foldings of tilable grids. Let $\mathcal{A}(\mathcal{T})$ denote the disjoint union of $\mathcal{A}'$ and $\mathcal{A}_\mathcal{T}$. Then $\mathcal{A}(\mathcal{T})$ has the desired property. □

As outline on page 47, the existence of the Σ-ACA $\mathcal{A}(\mathcal{T})$ implies the undecidability of the universality problem for Σ-ACAs:

Theorem 4.1.7 *Let Σ be an alphabet with at least two letters. Then there is no algorithm that on input of a Σ-ACA $\mathcal{A}$ decides whether it accepts all Σ-dags, i.e. whether $L(\mathcal{A}) = \mathbb{D}$.*

Proof. It is clearly sufficient to consider the case $\Sigma = \{a, b\}$. Let $\mathfrak{C}$ be a finite set of colors and $\mathcal{T}$ be a set of tiles. By Lemma 4.1.6, $\mathcal{A}(\mathcal{T})$ accepts all Σ-dags iff all grids allow a tiling. But this is equivalent to the existence of an infinite tiling which is undecidable. □

Since there is a Σ-ACA that accepts all Σ-dags, we get as an immediate

Corollary 4.1.8 *Let Σ be an alphabet with at least two letters. Then the equivalence of Σ-ACAs, i.e. the question whether $L(\mathcal{A}_1) = L(\mathcal{A}_2)$, is undecidable.*

By Corollary 4.1.8, the equivalence of two Σ-ACAs is undecidable. Rice's Theorem implies that for any Turing machine $\mathcal{M}$, the set of equivalent Turing machines is not recursive. This does not hold for Σ-ACAs in general: Let $L \subseteq \mathbb{D}$ be finite. Then the set of Σ-ACAs $\mathcal{A}$ with $L(\mathcal{A}) = L$ is recursive: Let $n := \max\{|V| \mid (V, E, \lambda) \in L\}$. Then, given a Σ-ACA $\mathcal{A}$, one can first check whether $L(\mathcal{A}) \cap \{(V, E, \lambda) \in \mathbb{D} \mid |V| \leq n\} = L$ since the set $\{(V, E, \lambda) \in \mathbb{D} \mid |V| \leq n\}$ is finite and $L(\mathcal{A})$ is recursive. In addition, one can easily construct from $\mathcal{A}$ a Σ-ACA $\mathcal{A}'$ such that $L(\mathcal{A}') = L(\mathcal{A}) \setminus \{(V, E, \lambda) \in \mathbb{D} \mid |V| \leq n\}$ (the Σ-ACA $\mathcal{A}'$ has to count the vertices up to n and accepts only if $\mathcal{A}$ accepts and there are at least $n+1$ nodes). Now, by Theorem 3.3.5, it can be checked whether $L(\mathcal{A}') = \emptyset$, i.e. whether $L(\mathcal{A}) = L$.

It is not clear whether there are other sets $L \subseteq \mathbb{D}$ such that the question whether $L(\mathcal{A}) = L$ can be decided.

By Example 2.1.6, there are Σ-ACAs that cannot be complemented. Hence, they are not equivalent to a deterministic one. Our next goal is to show that it is even undecidable whether a given ACA can be complemented or is equivalent to a determinstic ACA (Theorem 4.1.10).

Lemma 4.1.9 *Let $M \subseteq \mathbb{G}$ be such that for any $i \in \mathbb{N}^+$ there exist $k, \ell \in \mathbb{N}^+$ with $i \leq k$ and $1 < \ell$ such that $([i] \times [\ell], E, \lambda) \in M$. Let $\mathcal{A}$ be a Σ-ACA with $M \subseteq L(\mathcal{A})$. Then $L(\mathcal{A}) \not\subseteq \mathbb{G}$.*

Proof. Let $k \geq |Q_b| + 3$ and $1 < \ell$ such that $([k] \times [\ell], E, \lambda) \in M$. Then $k - 3$ is at least the number of states of the second process of $\mathcal{A}$. Since $\mathcal{A}$ accepts all elements of M, there is a successful run r of $\mathcal{A}$ on $([k] \times [\ell], E, \lambda)$. Since k is sufficiently large, there exist m, n with $1 < m < n < k$ such that $r(m, \ell) = r(n, \ell)$.

Now delete all vertices (m', ℓ) in $[k] \times [\ell]$ with $m < m' \leq n$, i.e. define P to be the set $[k] \times [\ell] \setminus \{(m', \ell) \mid m < m' \leq n\}$. Furthermore, let $E' := (E \cap P^2) \cup \{((m, \ell), (n+1, \ell))\}$. Then one can easily check that $(P, E', \lambda \restriction P)$ is a Σ-dag that does not belong to $\mathbb{G}$. We show that the restriction of the run r to P is a successful run of $\mathcal{A}$ on (P, E', λ): Note that the node $(n+1, \ell)$ is the only one from P whose set of lower neighbors in $([k] \times [\ell], E, \lambda)$ (where it equals $\{(n, \ell), (n + 1, \ell - 1)\}$) and in $(P, E', \lambda \restriction P)$ (where it equals $\{(m, \ell), (n + 1, \ell - 1)\}$) differ. But since $r(m, \ell) = r(n, \ell)$, this does not influence the run condition. Hence $(P, E', \lambda \restriction P)$ is accepted by $\mathcal{A}$, i.e. $L(\mathcal{A}) \not\subseteq \mathbb{G}$. □

Theorem 4.1.10 *Let Σ be an alphabet with at least two letters. Then there is no algorithm that on input of a Σ-ACA $\mathcal{A}$ decides any of the following questions:*
1. *Is $\mathbb{D} \setminus L(\mathcal{A})$ recognizable?*
2. *Is $\mathcal{A}$ equivalent with some deterministic Σ-ACA?*

Proof. Again, it is sufficient to consider the case $\Sigma = \{a, b\}$. Let $\mathcal{T}$ be a finite set of tiles and let $\mathcal{A}(\mathcal{T})$ be the Σ-ACA from Lemma 4.1.6, i.e. $\mathcal{A}(\mathcal{T})$ accepts a Σ-dag $t = (V, E, \lambda)$ iff
a) t is no folded grid, or
b) t is a folded grid that allows a tiling by $\mathcal{T}$.
Then $L := \mathbb{D} \setminus L(\mathcal{A}(\mathcal{T}))$ is the set of all folded grids that do not allow a tiling by $\mathcal{T}$. We show that L is recognizable iff $\mathcal{T}$ allows an infinite tiling:

If $\mathcal{T}$ allows an infinite tiling, L is empty and therefore trivially recognizable. Conversely, let $\mathcal{A}$ be a Σ-ACA that recognizes L. By contradiction, suppose that $\mathcal{T}$ does not allow an infinite tiling. Then, by Lemma 4.1.1, the set of tilable grids is not unbounded, i.e. there exist $k, \ell \in \mathbb{N}^+$ such that for any $k' \geq k$ and $\ell' \geq \ell$ the grid $[k'] \times [\ell']$ cannot be tiled. Thus, any folding of a grid $[k'] \times [\ell]$ with $k' \geq k$ belongs to L. Let $M := \{[k'] \times [\ell] \mid k' \geq k\}$. Then this set satisfies the condition of Lemma 4.1.9 and $M \subseteq L = L(\mathcal{A})$. Hence $L(\mathcal{A}) \not\subseteq \mathbb{G}$, contradicting $L(\mathcal{A}) = L \subseteq \mathbb{G}$.

This finishes the proof of the first statement since the existence of an infinite tiling and therefore the recognizability of $\mathbb{D} \setminus L(\mathcal{A}(\mathcal{T}))$ is undecidable.

Along the same line we can prove the second statement: If $\mathcal{A}(\mathcal{T})$ is equivalent with a deterministic Σ-ACA, $\mathbb{D} \setminus L(\mathcal{A}(\mathcal{T}))$ is recognizable since any deterministic Σ-ACA can be complemented. Hence $\mathbb{D} \setminus L(\mathcal{A}(\mathcal{T})) = \emptyset$ and therefore $\mathcal{T}$ allows an infinite tiling. Conversely, if $\mathcal{T}$ allows an infinite tiling, $\mathbb{D} \setminus L(\mathcal{A}(\mathcal{T})) = \emptyset$ implying $L(\mathcal{A}(\mathcal{T})) = \mathbb{D}$. But this set can be recognized deterministically, i.e. the ACA $\mathcal{A}(\mathcal{T})$ is equivalent with a deterministic one. □

Obviously, deterministic ACAs can be complemented. I do not know whether the inverse implication holds as well: Is any complementable Σ-ACA equivalent to a deterministic ACA?