Chapter 10

Kleene's Theorem

Theorem 9.3.8 characterizes the recognizable languages in a divisibility monoid with finitely many residuum functions using the concept of c-rationality which is a more restrictive notion than rationality. The aim of this section is to characterize those divisibility monoids that satisfy Kleene's Theorem: A divisibility monoid $(M, \cdot, 1)$ is *width-bounded* provided there exists $n \in \mathbb{N}$ with $w(\downarrow x, \leq) \leq n$ for any $x \in M$. Thus, a divisibility monoid is width-bounded if there is a uniform bound for the width of the lattices $\downarrow x$. Hence in the partial order $(M, \leq)$, bounded antichains have a uniformly bounded size. Note that a free monoid is width-bounded with $n = 1$ and that a direct product of two free monoids is not width-bounded. Hence a trace monoid is width-bounded iff it is free.

10.1 Rational monoids

Rational monoids are the main tool in our proof that any width-bounded divisibility monoid satisfies Kleene's Theorem. This concept was introduced by Sakarovitch [Sak87]. He showed that rational monoids satisfy Kleene's Theorem and considered closure properties of this class of monoids (cf. also [PS90] where the latter topic was extended). In this section, we recall some definitions and results from [Sak87] and prove a first statement concerning divisibility monoids.

Let $(M, \cdot, 1)$ be a monoid. A *generating system* of M is a pair (X, α) where X is a set and $\alpha : X^\star \to M$ is a surjective homomorphism. Then the *kernel of* α, i.e. the binary relation $\ker \alpha = \{(v, w) \in X^\star \times X^\star \mid \alpha(x) = \alpha(y)\}$, is a congruence relation on the free monoid $X^\star$.

An idempotent function $\beta : X^\star \to X^\star$ with $\ker \beta = \ker \alpha$ is a *description of* (X, α). We can think of $\beta(v)$ as a normal form of the word v. Note that (X, α) might have several descriptions. But for any such description β, $M \cong T^\star / \ker \beta$ since $\ker \beta = \ker \alpha$.

Let $(M, \cdot, 1)$ be a divisibility monoid. In Section 7.3, we defined the set $\mathcal{C}\ell$ to consist of all nonempty subsets of T of pairwise complementary elements that are bounded in $(M, \leq)$. Furthermore, α was defined to be the extension of the function $A \mapsto \sup(A)$ to a homomorphism from $\mathcal{C}\ell^\star$ onto M. Hence the tuple $(\mathcal{C}\ell, \alpha)$ is a generating system of the divisibility monoid $(M, \cdot, 1)$. Furthermore, we constructed an automaton $\mathcal{A}$ on the monoid $T^\star \times \mathcal{C}\ell^\star$ that computes the function $\mathrm{fnf} \circ \mathrm{nat} : T^\star \to \mathcal{C}\ell^\star$ by Theorem 7.3.6. The following proof uses this function to show that $\beta := \mathrm{fnf} \circ \alpha$ is a description of the generating system $(\mathcal{C}\ell, \alpha)$:

Lemma 10.1.1 *Let $(M, \cdot, 1)$ be a divisibility monoid. Then $\mathrm{fnf} \circ \alpha : \mathcal{C}\ell^\star \to \mathcal{C}\ell^\star$ is a description of $(\mathcal{C}\ell, \alpha)$.*

Proof. For any $A \in \mathcal{C}\ell$, choose some word $w_A \in T^\star$ with $\mathrm{nat}(w_A) = \alpha(A)$. Then there exists a homomorphism $\psi : \mathcal{C}\ell^\star \to T^\star$ that extends the mapping $A \mapsto w_A$. In addition, $\mathrm{nat} \circ \psi : \mathcal{C}\ell^\star \to M$ satisfies

$$\begin{aligned}
\mathrm{nat} \circ \psi(A_1 A_2 \dots A_n) &= \mathrm{nat}(w_{A_1} w_{A_2} \dots w_{A_n}) \\
&= \mathrm{nat}(w_{A_1}) \cdot \mathrm{nat}(w_{A_2}) \cdots \mathrm{nat}(w_{A_n}) \\
&= \alpha(A_1) \cdot \alpha(A_2) \cdots \alpha(A_n) \\
&= \alpha(A_1 A_2 \dots A_n),
\end{aligned}$$

i.e. $\alpha = \mathrm{nat} \circ \psi$. Hence $\beta = \mathrm{fnf} \circ \mathrm{nat} \circ \psi$.

It remains to show that β is idempotent and that $\ker \beta = \ker \alpha$: Since $\mathrm{fnf}(x)$ is the unique word in FNF with $\alpha(\mathrm{fnf}(x)) = x$, we have $\alpha \circ \mathrm{fnf} = \mathrm{id}_M$. Hence $\beta \circ \beta = \mathrm{fnf} \circ \alpha \circ \mathrm{fnf} \circ \alpha = \mathrm{fnf} \circ \mathrm{id}_M \circ \alpha = \beta$, i.e. β is idempotent. Now let $v, w \in \mathcal{C}\ell^\star$ with $\alpha(v) = \alpha(w)$. Then, clearly, $\beta(v) = \mathrm{fnf} \circ \alpha(v) = \mathrm{fnf} \circ \alpha(w) = \beta(w)$, i.e. $\ker(\alpha) \subseteq \ker(\beta)$. Conversely, $\beta(v) = \beta(w)$ implies $\alpha \circ \mathrm{fnf} \circ \alpha(v) = \alpha \circ \mathrm{fnf} \circ \alpha(w)$ and therefore $\ker \beta \subseteq \ker \alpha$ by $\alpha \circ \mathrm{fnf} = \mathrm{id}_M$. $\square$

A function $\beta : M \to N$ mapping one monoid into another can be seen as a subset of $M \times N$. Since this direct product is a monoid, we can speak of rational sets in $M \times N$. In this spirit, a function $\beta : M \to N$ is a *rational function* if it is a rational set in $M \times N$.

A monoid $(M, \cdot, 1)$ is a *rational monoid* if there exists a generating system (X, α) of M that has a rational description. Loosely speaking, a monoid is rational if there is a rational normal form function β that determines M. Let $\beta : X^\star \to X^\star$ be a rational description of the rational monoid M. Since the image of a rational set under a rational function is a rational set, the set $\beta(X^\star)$ is rational in the free monoid $X^\star$. Hence $M = \alpha \circ \beta(X^\star)$ is rational in M. Since any rational set in M is contained in a finitely generated submonoid of M, this implies that a rational monoid is finitely generated.

The key property of rational monoids that will be used in our considerations is that they satisfy Kleene's Theorem:

Theorem 10.1.2 ([Sak87, Theorem 4.1]) *Let M be a rational monoid and $L \subseteq M$. Then L is rational iff it is recognizable.*

Suppose the trace monoid $\mathbb{M}(\Sigma, D)$ is rational. Then it satisfies Kleene's Theorem implying that it is free. Since, conversely, any free monoid is rational, a trace monoid is rational iff it is free.

10.2 Width-bounded divisibility monoids

10.2.1 Width-bounded divisibility monoids are rational

In this section, we will show that the description $\mathrm{fnf} \circ \alpha$ of the generating system $(\mathcal{C}\ell, \alpha)$ for a width-bounded divisibility monoid is a rational function. To this purpose, we first show that the function $\mathrm{fnf} \circ \mathrm{nat}$ is rational. This is based on the following theorem that characterizes rational subsets in a monoid.

Theorem 10.2.1 ([EM65]) *Let M be a monoid. A set $L \subseteq M$ is rational iff it is the behavior of a finite automaton over M.*

Recall that an automaton is finite whenever its set of transitions is finite. Since the transitions of the automaton $\mathcal{A}$ from Theorem 7.3.6 are elements of the set $Q \times (T \times \mathcal{C}\ell_\varepsilon) \times Q$, and since the set $T \times \mathcal{C}\ell_\varepsilon$ is finite, it suffices to show that there are only finitely many reachable states. To this purpose, we show that the length of the monoid elements in reachable states is bounded. But first, we need the following lemma on the lattices $\downarrow x$ for $x \in M$. As known from traces, the width of these lattices is in general unbounded. Here we show that nevertheless the width of the join-irreducible elements is bounded by T:

Lemma 10.2.2 *Let $(M, \cdot, 1)$ be a divisibility monoid and $x \in M$. Then the width of $(\mathbb{J}(x), \leq)$ is at most $|T|$.*

Proof. Let $A \subseteq \mathbb{J}(x)$ be an antichain. Define

$$b := \sup\{y \in \mathbb{J}(x) \mid \neg\exists a \in A : a \leq y\}.$$

Since $\downarrow b \cap \mathbb{J}(x)$ equals $\{y \in \mathbb{J}(x) \mid \neg\exists a \in A : a \leq y\}$ and since A is an antichain, $\downarrow b \cap \mathbb{J}(x)$ is the set of minimal elements of the partially ordered set $\mathbb{J}(x) \setminus \downarrow b$. By Lemma 7.3.1, $|A|$ equals the number of minimal elements of $\mathbb{J}([b, x])$. Since $[b, x]$ and $\downarrow b^{-1}x$ are order isomorphic by Lemma 7.1.1, $|A|$ is the number of minimal elements of $\mathbb{J}(b^{-1}x)$, i.e. of elements $t \in T$ with $t \leq b^{-1}x$. Hence $|A| \leq |T|$. □

Now we can bound the number of reachable states in the automaton $\mathcal{A}$.

Lemma 10.2.3 *Let $(M, \cdot, 1)$ be a width-bounded divisibility monoid such that $w(\downarrow x, \leq) \leq n$ for any $x \in M$. Let $x, y \in M$ with $|\mathrm{fnf}(xy)| = |\mathrm{fnf}(x)|$. Then $|y| < 2(n+1)|T|$.*

Proof. By contradiction, assume $|y| \geq 2(n+1)|T|$. Since $x \leq xy$, the set $\mathbb{J}(x)$ is an ideal in $(\mathbb{J}(xy), \leq)$. Hence, for $v \in \mathbb{J}(x)$ and $w \in \mathbb{J}(xy)$, it holds $w \not\leq v$. The size of $\mathbb{J}(x)$ equals the length of $(\downarrow x, \leq)$ and therefore of x and similarly for xy. Hence $\mathbb{J}(xy) \setminus \mathbb{J}(x)$ contains at least $2(n+1)|T|$ elements. By Lemma 10.2.2, $\mathbb{J}(xy) \setminus \mathbb{J}(x)$ has width at most $|T|$. Hence the elements of $\mathbb{J}(xy) \setminus \mathbb{J}(x)$ occupy at least $2(n+1)$ different heights, i.e. there are natural numbers $0 \leq n_1 < n_2 \cdots < n_{2(n+1)}$ such that there exists $w_i \in \mathbb{J}(xy) \setminus \mathbb{J}(x)$ with $h(w_i, \mathbb{J}(xy)) = n_i$ for $1 \leq i \leq 2(n+1)$. Since $|\mathrm{fnf}(xy)| = |\mathrm{fnf}(x)|$, the partially ordered sets $\mathbb{J}(xy)$ and $\mathbb{J}(x)$ have the same length by Lemma 7.3.2. Hence, for $1 \leq i \leq 2(n+1)$ there exists $v_i \in \mathbb{J}(x)$ with $h(v_i, \mathbb{J}(xy)) = n_i$. Since $h(w_i, \mathbb{J}(xy)) \leq h(v_j, \mathbb{J}(xy))$ for $1 \leq i \leq n \leq j \leq 2(n+1)$, the elements from $\{w_i \mid 1 \leq i \leq n+1\}$ and $\{v_j \mid n+1 \leq j \leq 2(n+1)\}$ are mutually incomparable. Then $I(i,j) := \downarrow\{w_1, w_2, \ldots, w_i, v_{n+1}, v_{n+2}, \ldots, v_{n+1+j}\}$ is a finitely generated ideal in $(\mathbb{J}(xy), \leq)$. Note that $w_j \not\leq w_i$ for $1 \leq i < j \leq n+1$ and similarly $v_j \not\leq v_i$ for $n+1 \leq i < j \leq 2(n+1)$. Hence the ideals $I(i, n-i)$ for $1 \leq i \leq n+1$ are pairwise incomparable, i.e. $(\mathbb{H}(\mathbb{J}(xy), \leq), \subseteq)$ contains an antichain of $n+1$ elements. Since $(\downarrow xy, \leq) \cong (\mathbb{H}(\mathbb{J}(xy), \leq), \subseteq)$, this contradicts our assumption. $\square$

The proof of the following theorem is based on the fact that the description $\mathrm{fnf} \circ \alpha$ of the generating system $(\mathcal{C}\ell, \alpha)$ for a width-bounded divisibility monoid is rational:

Theorem 10.2.4 *Any width-bounded divisibility monoid is a rational monoid.*

Proof. Let M be a width-bounded divisibility monoid. By Theorem 7.3.6, the automaton $\mathcal{A}$ computes the function $\mathrm{fnf} \circ \mathrm{nat} : T^\star \to \mathcal{C}\ell^\star$. To show that this is rational, it remains to prove that the number of reachable states in $\mathcal{A}$ is finite (since the transitions are labeled by the finite set $T \times \mathcal{C}\ell_\varepsilon$). Let (z, C) be a reachable state of $\mathcal{A}$. Then, by Lemma 7.3.5, there exists $x \in M$ with $|\mathrm{fnf}(x)| = |\mathrm{fnf}(xz)|$. Hence, by Lemma 10.2.3, the length of z is bounded by $2(n+1)|T|$ where n is the global bound for the size of bounded antichains in $(M, \leq)$. Since $\mathcal{C}\ell$ is finite, this implies that there are only finitely many reachable states in $\mathcal{A}$.

Recall that $(\mathcal{C}\ell, \alpha)$ is a generating system of M. By Lemma 10.1.1, the function $\mathrm{fnf} \circ \alpha : \mathcal{C}\ell^\star \to \mathcal{C}\ell^\star$ is a description of $(\mathcal{C}\ell, \alpha)$. To show that this description is rational, consider the homomorphism $\psi : \mathcal{C}\ell^\star \to T^\star$ defined in the proof of Lemma 10.1.1, where we also showed $\alpha = \mathrm{nat} \circ \psi$ and therefore $\beta = \mathrm{fnf} \circ \mathrm{nat} \circ \psi$. Since ψ is a homomorphism, it is a rational relation from $\mathcal{C}\ell^\star$ into $T^\star$, i.e. β splits into two rational relations $\mathcal{C}\ell^\star \to T^\star$ and $T^\star \to \mathcal{C}\ell^\star$. Since $T^\star$ is a free monoid, by [EM65] (cf. [Sak87, Proposition A.16]), β is rational. □

Remark. By [Sak87, Theorem 4.1], Kleene's Theorem holds in any rational monoid. Thus, the theorem above implies that in a width-bounded divisibility monoid the rational and the recognizable sets coincide. There is an alternative proof of this weaker result that follows the line of the proof of Theorem 9.3.8: Let $(M, \cdot, 1)$ be a width-bounded divisibility monoid $(M, \cdot, 1)$ with $w(\downarrow x, \leq) \leq n$ for any $x \in M$. Then one shows that its monoid of residuum functions $\mathbb{R}_M$ has at most $|T|^{n+1} - 1 + |T|^{(n+1)(|T|^{n+1}-1)}$ elements. Hence any such monoid has finitely many residuum functions. The crucial point then is to show that the rank of X is bounded by $2n$ for any $X \subseteq T^\star$.

10.2.2 Rational divisibility monoids are width-bounded

Our next goal is to show that the width-boundedness is not only sufficient but also necessary for Kleene's Theorem to hold. We start with two lattice-theoretic lemmata.

Lemma 10.2.5 *Let $(P, \leq)$ be a partially ordered set, $M, N \subseteq P$ sets with $n-1$ elements each such that any $m \in M$ is incomparable with any $n \in N$. Then there exists a semilattice embedding of $[n-1] \times [n-1]$ into $(\mathbb{H}_f(P), \subseteq)$. If $(P, \leq)$ is finite, this embedding can be chosen to preserve infima, too.*

Proof. Let $M = \{m_1, m_2 \ldots, m_{n-1}\}$ and $N = \{n_1, n_2 \ldots, n_{n-1}\}$ be linear extensions of $(M, \leq)$ and $(N, \leq)$, i.e. $m_i \leq m_j$ or $n_i \leq n_j$ implies $i \leq j$. Then $I(i,j) := \downarrow\{m_1, \ldots, m_i, n_1, \ldots, n_j\}$ is a finitely generated ideal and therefore an element of $\mathbb{H}_f(P, \leq)$. Furthermore, $(\{I(i,j) \mid 1 \leq i \leq n-1, 1 \leq j \leq n-1\}, \subseteq)$ is the desired subposet of $\mathbb{H}_f(P, \leq)$. □

Next, we want to prove that any distributive lattice of sufficient width contains a large grid. Recall that $R_{n+1}(6^n)$ is a Ramsey number (cf. Ramsey's Theorem on page 61).

Theorem 10.2.6 *Let $(L, \leq)$ be a finite distributive lattice with $w(L) \geq R_{n+1}(6^n)$. Then there exists a lattice embedding of $[n-1] \times [n-1]$ into $(L, \leq)$.*

Proof. First we consider the case $w(\mathbb{J}(L)) \geq 2n$. Then there exists an antichain $A \subseteq \mathbb{J}(L)$ containing $2n$ elements. Let M and N be disjoint subsets of A of size n. Then Lemma 10.2.5 implies the statement.

Now assume $w(\mathbb{J}(L)) = k < 2n$. By Dilworth's Theorem [Dil50], there are chains $C_1, C_2, \ldots, C_k \subseteq \mathbb{J}(L)$ with $\mathbb{J}(L) = \bigcup_{i=1,\ldots k} C_i$. For $x \in L$ and $1 \leq \ell \leq k$ let $\partial_\ell(x)$ denote the maximal element of C_ℓ below x if it exists, and $\perp$ otherwise. To ease the notations in this proof, we will consider $\perp$ as an additional element of $\mathbb{J}(L)$ which is minimal. Since L is distributive, $x = \bigvee_{1 \leq i \leq k} \partial_i(x)$ for any $x \in L$.

Since $w(L) \geq R_{n+1}(6^n)$, there is an antichain $A = \{x_1, x_2, \ldots, x_m\}$ in L with $m \geq R_{n+1}(6^n)$. Now we define a mapping $g_{i,j} : \{1, 2, \ldots, k\} \to \{<, =, >\}$ for $1 \leq i < j \leq m$ by $g_{i,j}(\ell) = \theta$ iff $\partial_\ell(x_i)\theta\partial_\ell(x_j)$ (with $\theta \in \{<, =, >\}$). For $x_i \neq x_j$, we define $g(x_i, x_j) = g_{\min\{i,j\},\max\{i,j\}}$. Thus, g maps the two-element subsets of A into $\{<, =, >\}^{\{1,2,\ldots,k\}}$. Since this set contains at most $3^{2n} = 6^n$ elements and since $m > R_{n+1}(6^n)$, we can assume $g(x_i, x_j) = g(x_{i'}, x_{j'}) =: f$ for $i, j, i', j' \in \{1, 2, \ldots, n+1\}$ with $i \neq j$ and $i' \neq j'$. Then $f(\ell_1) \neq$"$=$" for some $1 \leq \ell_1 \leq k$ since otherwise $x_1 = x_2$. Similarly, there is an index $1 \leq \ell_2 \leq k$ with $f(\ell_2) \notin \{=, f(\ell_1)\}$ since otherwise x_1 and x_2 are comparable. Without loss of generality, we assume $f(1) =$"$<$" and $f(2) =$"$>$". Then $\partial_1(x_1) < \partial_1(x_2) < \ldots \partial_1(x_{n+1})$ and $\partial_2(x_1) > \partial_2(x_2) > \ldots \partial_2(x_{n+1})$. Thus, $C_j := \{\partial_j(x_i) \mid 1 < i < n+1\}$ for $j = 1, 2$ is a chain in $\mathbb{J}(L)$ containing $n-1$ elements.

Let $1 < i < n+1$ with $\partial_1(x_i) \geq \partial_2(x_i)$. Then $x_{i+1} \geq \partial_1(x_{i+1}) > \partial_1(x_i) \geq \partial_2(x_i) > \partial_2(x_{i+1})$ and $\partial_2(x_i), \partial_2(x_{i+1}) \in C_2$. But this contradicts the definition of $\partial_2(x_{i+1})$ as the maximal element of C_2 below x_{i+1}. Symmetrically, we can argue if $\partial_1(x_i) \leq \partial_2(x_i)$ (with x_{i-1} in place of x_{i+1}). Thus, $\partial_1(x_i)$ and $\partial_2(x_i)$ are incomparable for $1 < i < n+1$.

Now let $1 < i < j < n+1$ with $\partial_1(x_j) \geq \partial_2(x_i)$. Then $\partial_2(x_i) > \partial_2(x_j)$ since $i < j$, i.e. $\partial_1(x_j) > \partial_2(x_j)$, a contradiction to what we showed above. Similarly, we can argue in the cases $\partial_1(x_j) \leq \partial_2(x_i)$, $\partial_1(x_i) \geq \partial_2(x_j)$ and $\partial_1(x_i) \leq \partial_2(x_j)$.

Thus, we found two chains C_1 and C_2 in $\mathbb{J}(L)$ of size $n-1$ whose elements are mutually incomparable. Now Lemma 10.2.5 implies that $[n-1] \times [n-1]$ can be lattice embedded into $(\mathbb{H}(\mathbb{J}(L)), \subseteq) \cong L$. □

The following lemma implies that the free commutative monoid with two generators can be embedded into a divisibility monoid if the size of bounded antichains is unbounded.

Lemma 10.2.7 *Let $(M, \cdot, 1)$ be a divisibility monoid with finitely many residuum functions. Let $z \in M$ with $w(\downarrow z) \geq R_{n+2}(6^{n+1})$ where $n = R_3(|\mathbb{R}_M|) + 1$. Then there exist $x, y \in M \setminus \{1\}$ such that $x \,\mathbf{co}\, y$, $r_y(x) = x$ and $r_x(y) = y$.*

Proof. By Theorem 10.2.6, there is a lattice embedding $\eta : [n-1]^2 \to \downarrow z$. By cancellation, we may assume $\eta(0, 0) = 1$. For $0 \leq i \leq n-2$ there is $x_i \in M \setminus \{1\}$ with $\eta(i, 0) \cdot x_i = \eta(i+1, 0)$. By Lemma 9.2.4, there are $0 \leq i < j \leq n-1$ with $r_{x_i x_{i+1} \cdots x_{j-1}} \subseteq \mathrm{id}_M$. Furthermore, there are $y_\ell \in M \setminus \{1\}$ with $\eta(i, \ell) \cdot y_\ell = \eta(i, \ell+1)$. Using Lemma 9.2.4 again, there are $0 \leq k < \ell \leq n-1$ with $r_{y_k y_{k+1} \cdots y_{\ell-1}} \subseteq \mathrm{id}_M$.

Let $x := \eta(i,k)^{-1}\eta(j,k)$ and $y := \eta(i,k)^{-1}\eta(i,\ell) = y_k y_{k+1}\cdots y_{\ell-1}$. Then $r_y \subseteq \mathrm{id}_M$.

To show $r_x \subseteq \mathrm{id}_M$, note that $\eta(i,0) = \eta(i,k) \wedge \eta(j,0)$ since $i < j$. Hence, by Lemma 7.1.1, $1 = \eta(i,0)^{-1}\eta(i,0) = \eta(i,0)^{-1}\eta(i,k) \wedge \eta(i,0)^{-1}\eta(j,0)$, i.e. $\eta(i,0)^{-1}\eta(i,k)$ and $\eta(i,0)^{-1}\eta(j,0)$ are complementary.

Similarly, we get $\eta(i,k) \vee \eta(j,0) = \eta(j,k)$ since $i < j$ and therefore

$$\begin{aligned}\eta(i,0)^{-1}\eta(i,k) \vee \eta(i,0)^{-1}\eta(j,0) &= \eta(i,0)^{-1}\eta(j,k)\\ &= \eta(i,0)^{-1}\eta(i,k)\ \eta(i,k)^{-1}\eta(j,k).\end{aligned}$$

Thus $\eta(i,0)^{-1}\eta(j,0) \uparrow \eta(i,0)^{-1}\eta(i,k) = \eta(i,k)^{-1}\eta(j,k) = x$. Since the residuum function of $x_i x_{i+1}\cdots x_{j-1} = \eta(i,0)^{-1}\eta(j,0)$ is contained in the identity, Lemma 7.2.6 indeed implies $r_x \subseteq \mathrm{id}_M$.

It remains to show $x \,\mathbf{co}\, y$: Since $\eta(j,k), \eta(i,\ell) \le \eta(j,\ell)$, the elements x and y are bounded in $(M,\le)$. Furthermore, $\eta(i,\ell) \wedge \eta(j,\ell) = \eta(i,k)$ implies $x \wedge y = 1$. □

Now we can characterize the divisibility monoids that satisfy Kleene's Theorem.

Theorem 10.2.8 *Let $(M,\cdot,1)$ be a divisibility monoid with finitely many residuum functions. Then the following are equivalent*

1. *M is width-bounded,*
2. *M is rational, and*
3. *any set $L \subseteq M$ is rational iff it is recognizable.*

Proof. The implication $1 \Rightarrow 2$ follows from Theorem 10.2.4, and the implication $2 \Rightarrow 3$ from [Sak87, Theorem 4.1]. Now assume M not to be width-bounded. Then, by Lemma 10.2.7, there are $x, y \in M \setminus \{1\}$ such that $x \,\mathbf{co}\, y$, $r_x(y) = y$ and $r_y(x) = x$. Hence we can embed the monoid $(\mathbb{N}\times\mathbb{N}, +, (1,1))$ into M (extending the mapping $(1,0) \mapsto x$ and $(0,1) \mapsto y$ to a homomorphism). Since $\{(i,i) \mid i \in \mathbb{N}\}$ is rational but not recognizable in $(\mathbb{N}\times\mathbb{N}, +, (0,0))$, its image is so in M. Hence M does not satisfy Kleene's Theorem, i.e. the implication $3 \Rightarrow 1$ is shown. □

Remark. Note that the assumption on M to have finitely many residuum functions is necessary for the implication $3 \Rightarrow 1$, only. On the other hand, the implications $1 \Rightarrow 2 \Rightarrow 3$ can be shown without this assumption. It is not clear whether the other implications, in particular that any rational divisibility monoid is width-bounded can be shown without this assumption.

Above, we used Sakarovitch's result that rational monoids are Kleene monoids. Together with Peletier, he showed that the converse is false, i.e., that there are

Kleene monoids that are not rational [PS90]. Rupert showed that any commutative Kleene monoid is rational [Rup91]. Similarly, we showed above that any Kleene divisibility monoid is rational.