Chapter 11

Monadic second order logic

11.1 Two Büchi-type theorems

Büchi showed that the monadic second order theory of the linearly ordered set $(\omega, \leq)$ is decidable. To achieve this goal, he used automata. In the course of these considerations it was shown that a language in a free finitely generated monoid is recognizable iff it is monadically axiomatizable. In computer science, this latter result and its extension to infinite words are often referred to as "Büchi's Theorem" while in logic it denotes the decidability of the monadic theory of ω. Here, I understand it in this second meaning, i.e. it is the aim of this section to show that certain monadic theories associated to a divisibility monoid are decidable. In particular, it will be shown that the monadic theory $\mathrm{MTh}(\{(\mathbb{J}(\downarrow m), \leq) \mid m \in M\})$ is decidable for any divisibility monoid with finitely many residuum functions.

Let $(L, \leq)$ be a finite distributive lattice. Let $x, y \in L$ with $x \longrightarrow\!\!\!< y$. Then there exists a uniquely determined join-irreducible element $z \in \mathbb{J}(L)$ such that $z \leq y$ and z is incomparable with x. We denote this element by $\mathrm{prim}(x, y)$. Then $x \vee \mathrm{prim}(x, y) = y$.

Lemma 11.1.1 *Let $(M, \cdot, 1)$ be a divisibility monoid. Furthermore, let $s, t \in T$ and $u, v \in M$. Then $\mathrm{prim}(u, us)$ and $\mathrm{prim}(usv, usvt)$ are incomparable iff there exist $x_1, x_2 \in M$ and $s' \in T$ such that*

$$sv = x_1 s' x_2, s' = r_{x_1}(s) \text{ and } t \in \mathrm{im}(r_{s'x_2}).$$

The situation of the lemma is depicted by Figure 11.1.

Proof. First, assume $\mathrm{prim}(u, us)$ and $\mathrm{prim}(usv, usvt) =: b$ to be incomparable. Since b is join-irreducible, there is a uniquely determined element $a \in M$ with $a \longrightarrow\!\!\!< b$. Assume $us \leq uva$. Then $\mathrm{prim}(u, us) \leq u \vee \mathrm{prim}(u, us) = us \leq a < b$, contradicting $\mathrm{prim}(u, us) \parallel b$. Hence $us \not\leq u \vee a$. Furthermore, $u \vee a < us \vee a$ for otherwise $us \leq us \vee a = u \vee a$ contradicting what we just showed. Hence

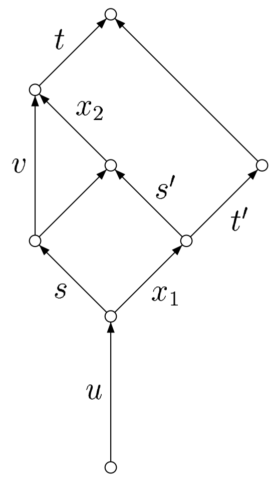

Figure 11.1: cf. Lemma 11.1.1

$u{\vee}a \mathrel{-\!\!<} us{\vee}a$, i.e. there exists $s' \in T$ with $(u{\vee}a){\cdot}s' = us{\vee}a$. Let $x_1 := u^{-1}(u{\vee}a)$, i.e. $u \vee a = ux_1$. Then $ux_1s' = \sup(u, us, a) = ux_1 \vee us$ implying $s' = x_1 \vee s$. On the other hand, $us \not\leq u \vee a = ax_1$, i.e. $s \not\leq x_1$ implies $s \wedge x_1 = 1$ since $s \in T$. Hence $x_1 \,\mathbf{co}\, s$ and therefore $s' = s \uparrow x_1 = r_{x_1}(s)$.

Next we show $u{\vee}a \mathrel{-\!\!<} u{\vee}b$. Clearly, $a \leq b$ implies $u{\vee}a \leq u{\vee}b$. Assume them to be equal. Since b is join-irreducible and above a, this implies $b \leq u \leq usv$. But this contradicts $b \vee usv = \mathrm{prim}(usv, usvt) \vee usv = usvt > usv$. Since $a \mathrel{-\!\!<} b$ we get $u \vee a \mathrel{-\!\!<} u \vee b$. Hence there exists $t' \in T$ with $u \vee a \cdot t' = u \vee b$.

Let $x_2 \in M$ be given by $(us \vee a) \cdot x_2 = usv$. It remains to show that we have $t = t' \uparrow s'x_2$: First note that $ux_1t' = u \vee b$ and

$$ux_1s'x_2 = (u \vee a)s'x_2 = (us \vee a)x_2 = usv.$$

Hence $ux_1t'{\vee}ux_1s'x_2 = u{\vee}b{\vee}usv = usvt = ux_1s'x_2t$. This implies $t'{\vee}s'x_2 = s'x_2t$ and therefore in particular $s'x_2 \mathrel{-\!\!<} t' \vee s'x_2$. Since $t' \in T$, this implies $t' \,\mathbf{co}\, s'x_2$ and $t = t' \uparrow s'x_2$.

Conversely, let $s', t' \in T$ and $x_1, x_2 \in M$ such that we have $sv = x_1s'x_2$, $s' = s \uparrow x_1$ and $t = t' \uparrow s'x_2$. Then $s \vee x_1 = x_1s'$ implying $us \vee ux_1 = ux_1s'$. Now $ux_1 \vee \mathrm{prim}(u, us) = ux_1 \vee u \vee \mathrm{prim}(u, us) = ux_1 \vee us = ux_1s'$ follows.

Similarly, $t' \vee s'x_2 = s'x_2t$ and therefore $ux_1t' \vee ux_1s'x_2 = ux_1s'x_2t$ or, since $ux_1t'x_2 = usv$, $ux_1t' \vee usv = usvt$. On the other hand, we have $usv \vee ux_1t' = ux_1s'x_2{\wedge}ux_1t' = ux_1(s'x_2{\wedge}t') = ux_1$ since $s'x_2 \,\mathbf{co}\, t'$. Hence the two prime intervals (ux_1, ux_1t') and $(usv, usvt)$ are transposed. Thus we get $ux_1 \vee \mathrm{prim}(usv, usvt) = ux_1 \vee \mathrm{prim}(ux_1, ux_1t') = ux_1t'$.

Since t' and $s'x_2$ are complementary, t' and s' are in particular incomparable. Hence so are ux_1s' and ux_1t'. Since, as we saw above, $ux_1 \vee \text{prim}(u, us) = ux_1s'$ and $ux_1 \vee \text{prim}(usv, usvt) = ux_1t'$, $\text{prim}(u, us)$ and $\text{prim}(usv, usvt)$ are incomparable. □

Let $w \in T^\star$ be a word. Then w determines the monoid element $x = \text{nat}(w) \in M$ and therefore the partial order $(\mathbb{J}(x), \leq)$. At the same time, w "is" a T-labeled linear order. The next lemma implies that the theory of $(\mathbb{J}(x), \leq)$ can be interpreted in the theory of the linear order w:

Lemma 11.1.2 *Let M be a divisibility monoid with finitely many residuum functions $\mathbb{R}_M$. Then there exists a monadic formula* less *over the signature $\{\leq, \lambda\}$ with two free elementary variables such that for any $w \in T^\star$:*

$$(\mathbb{J}(\downarrow[w]), <) \cong (\text{dom}(w), \{(x, y) \in (\text{dom}(w))^2 \mid w \models \text{less}(x, y)\}).$$

Proof. For $c \in \mathbb{R}_M$ let L_c denote the set of all $x \in M$ with $r_x = c$. Then, for $s, t \in T$, we have

$$\begin{aligned} M_{s,t} &:= \{x_1s'x_2 \mid x_1, x_2 \in M, s' \in T \text{ such that } s' = r_{x_1}(s) \text{ and } t \in \text{im}(r_{s'x_2})\} \\ &= \bigcup_{s' \in T} (\bigcup \{L_c \mid c \in \mathbb{R}_M, c(s') = s\} \cdot s' \cdot \bigcup \{L_{r_x} \mid x \in M, t \in \text{im}(r_{s'x})\}). \end{aligned}$$

Since M has finitely many residuum functions and $x \mapsto r_x$ is a monoid antihomomorphism, this set is recognizable by Corollary 9.2.2. Hence $\{w \in T^\star \mid [w] \in M_{s,t}\}$ is recognizable in $T^\star$ and therefore axiomatizable by a monadic sentence $\varphi_{s,t}$. Now we define

$$\text{less}(x, y) := \bigwedge_{s,t \in T} (\lambda(x) = s \wedge \lambda(y) = t \wedge x < y \wedge \neg\varphi'_{s,t})$$

where $\varphi'_{s,t}$ is the restriction of $\varphi_{s,t}$ to the positions between x and y, i.e. to the set $\{z \in \text{dom}(w) \mid x \leq z < y\}$. Now the lemma follows easily by the preceding lemma. □

Thus, indeed, the monadic theory of $(\mathbb{J}(\text{nat}(w)), \leq)$ can be interpreted in the monadic theory of the linear order w. In addition, the monadic theory of all linear orders in $T^\star$ is decidable. Hence the monadic theory of $\{(\mathbb{J}(x), \leq) \mid x \in M\}$ is decidabe:

Theorem 11.1.3 *Let $(M, \cdot, 1)$ be a divisibility monoid with finitely many residuum functions $\mathbb{R}_M$. Then the monadic theory* $\text{MTh}(\{(\mathbb{J}(\downarrow m), <) \mid m \in M\})$ *is decidable.*

Proof. Let φ be a monadic sentence over the signature $\{<\}$. In φ, replace any subformula of the form $x < y$ by $\mathrm{less}(x, y)$ and denote the resulting sentence by $\overline{\varphi}$. Then, for any $w \in T^\star$, we have $w \models \overline{\varphi}$ iff $(\mathbb{J}(\downarrow[w]), <) \models \varphi$. Since the monadic theory of the words over T is decidable, the result follows. $\square$

By the theorem above, the monadic theory of $\{(\mathbb{J}(\downarrow m), \leq) \mid m \in M\}$ is decidable. The union of all these sets is $\mathbb{J}(M)$, the set of join-irreducibles in $(M, \leq)$. The following theorem shows that the monadic theory of this set is not decidable in general:

Theorem 11.1.4 *Let (Σ, D) be a finite dependence alphabet. Then the monadic theory of $(\mathbb{J}(\mathbb{M}(\Sigma, D)), \leq)$ is decidable iff D is transitive.*

Proof. Let D be transitive. Then $\mathbb{J}(\mathbb{M}(\Sigma, D), \leq)$ is the disjoint union of trees of the form $(\{1, 2, \ldots, k\}^\star, \leq)$. Since the monadic theory of these uniformly branching trees is decidable [Rab69], so is the monadic theory of their disjoint union [She75].

On the other hand, suppose D not to be transitive. Then there are $a, b, c \in \Sigma$ with $(a, b), (b, c) \in D$ and $(a, c) \notin D$. We show how to encode an undirected graph (V, E) into two antichains A and B of $\mathbb{J}(\mathbb{M}(\Sigma, D))$: Suppose $V = \{1, 2, \ldots, n\}$. The vertices are represented by the elements of the set $A := \{a^k c^k b \mid 1 \leq k \leq n\}$. Furthermore, the edges of the graph (V, E) correspond to the elements of the antichain $B := \{a^i c^j b \mid (i, j) \in E\}$. Then, for any "vertices" $x, y \in A$, there is an edge in the graph (V, E) iff there exist $x', y' \in \mathbb{J}(\mathbb{M}(\Sigma, D))$ and $z \in B$ such that $x' \multimap x, z$ and $y' \multimap y, z$. Since this can be expressed by an elementary formula, we can reduce the elementary theory of graphs to the monadic antichain theory of $(\mathbb{M}(\Sigma, D), \leq)$. $\square$

Again, by Theorem 11.1.3, the monadic theory $\mathrm{MTh}\{\mathbb{J}(\downarrow x, \leq) \mid x \in M\}$ is decidable for any divisibility monoid with finitely many residuum functions. This does not imply that the monadic theory $\mathrm{MTh}\{(\downarrow x, \leq) \mid x \in M\}$ is decidable. A counterexample is provided by the free commutative monoid with two generators since this monoid contains, for any $n \in \mathbb{N}$, an element x such that $(\downarrow x, \leq)$ is the grid $([n]^2, \leq)$. We will show that these grids are the only reason for the undecidability.

To this aim, we first show that for a given divisibility monoid $(M, \cdot, 1)$ with finitely many residuum functions, the set of lattices $(\downarrow m, \leq)$ for $m \in M$ is finitely axiomatizable in monadic second order logic (Corollary 11.1.7).

Let Σ be a finite alphabet and consider the elementary logic that is appropriate to reason on Σ-labeled partially ordered sets. Furthermore, we deal with pomsets without autoconcurrency, only, i.e. we consider structures $t = (V, \leq, \lambda)$ where

$(V, \leq)$ is a finite partially ordered set and $\lambda : V \to \Sigma$ is a mapping such that $\lambda^{-1}(a)$ is linearly ordered in $(V, \leq)$. In this setting, one can write down a formula φ with two free variables x and y such that

$$\varphi^t = \{(x, y) \in E^2 \mid t \models \varphi(x, y)\}$$

is a linear extension of $\leq$. For traces over (Σ, D) this was shown in [EM96]. For Σ-labeled partially ordered sets that are associated to the computations of stably concurrent automata, it has been observed independently in [DK96]. The most compact formula that defines a linear order in pomsets without autoconcurrency can be found in [DM97]. They consider traces only. Nonetheless, following their argumentation verbatim, one can easily see that their formula defines a linear order extension of the partial order of any pomset without autoconcurrency. Knowing this, the following lemma is an immediate reformulation:

Lemma 11.1.5 *There exists a monadic formula* $\text{lin}(x, y, C_1, \ldots, C_m)$ *satisfying: For any finite partial order* $(P, \leq)$ *of with at most* n *and any chains* $C_i \subseteq P$ *for* $1 \leq i \leq m$ *such that* $P = \bigcup_{1 \leq i \leq m} C_i$ *and* $C_i \cap C_j = \emptyset$ *for* $1 \leq i < j \leq m$*, the relation*

$$\text{lin}^{(P, \leq, C_1, \ldots, C_m)} = \{(x, y) \in P^2 \mid (P, \leq) \models \text{lin}(x, y, C_1, \ldots, C_m)\}$$

is a linear order extending $\leq$.

Theorem 11.1.6 *Let* $(M, \cdot, 1)$ *be a divisibility monoid with finitely many residuum functions. There exists a monadic sentence* $\overline{\varphi}$ *such that for any finite partial order* $(P, \leq)$*:*

$$(P, \leq) \models \overline{\varphi} \iff \text{there exists } x \in M \text{ with } \mathbb{J}(\downarrow x, \leq) \cong (P, \leq).$$

Proof. Let $m = |T|$ denote the number of irreducible elements of the monoid $(M, \cdot, 1)$. By Lemma 11.1.2, there exists a formula less that defines the partially ordered set $\mathbb{J}(\downarrow[w], <)$ inside the word $w \in T^\star$. In this formula, replace any subformula of the form $\lambda(z) = t$ by $z \in M_t$ and any subformula of the form $z \leq z'$ by $\text{lin}(z, z', C_1, C_2, \ldots, C_m)$. The result is denoted by less$'$. Now let $\varphi(C_1, C_2, \ldots, C_m)$ denote the following formula

$$\begin{aligned} \exists_{t \in T} M_t \quad (\quad & \bigcup_{t \in T} M_t = \text{ everything} \wedge \\ & M_s \cap M_t = \emptyset \text{ for } s \neq t \wedge \\ & \forall x, y (x < y \leftrightarrow \text{less}'(x, y, C_1, \ldots, C_m)) \\ ). \quad & \end{aligned}$$

Let $x \in M$ and $(C_i)_{1 \leq i \leq m}$ be a tuple of mutually disjoint chains whose union equals $\mathbb{J}(\downarrow x, \leq)$. For simplicity, let $(P, \leq) := \mathbb{J}(\downarrow x, \leq)$. Then $\text{lin}^{(P, \leq, C_1, \ldots, C_m)}$

defines a linear order that extends $\leq$. This linear order defines a maximal chain in the lattice $(\downarrow x, \leq)$ which corresponds naturally to a word $w \in T^\star$ with $\mathrm{nat}(w) = x$. Now let M_t be the set of positions in w that are labeled by the irreducible element $t \in T$. Then the sets M_t satisfy the first two conditions of the formula φ. Furthermore, by Lemma 11.1.2, the last statement holds as well. Hence $(P, \leq) \models \varphi(C_1, \ldots, C_m)$. On the contrary, let $(P, \leq)$ be a finite partial order, C_i mutually disjoint chains whose union is P such that $(P, \leq) \models \varphi(C_1, \ldots, C_m)$. Let $P = (x_1, x_2, \ldots, x_k)$ be the enumeration of P that is completely defined by $(P, \leq) \models \mathrm{lin}(x_i, x_{i+1}, C_1, \ldots, C_m)$. Now consider the word $w = t_1 t_2 \ldots t_k$ with $x_i \in M_{t_i}$ for all i. Due to the construction of less$'$ from less and Lemma 11.1.2, $(P, \leq) \cong (\mathbb{J}(\downarrow(\mathrm{nat}(w)), \leq))$. Hence we found a monoid element $x = \mathrm{nat}(w)$ such that $(P, \leq) \cong (\mathbb{J}(\downarrow m, \leq)$.

Finally, let $\overline{\varphi}$ denote the formula

$$\begin{aligned} \exists_{1 \leq i \leq m} C_i \quad (\quad & \bigcup_{1 \leq i \leq m} C_i = \text{ everything} \wedge \\ & C_i \cap C_j = \emptyset \text{ for } 1 \leq i < j \leq m \wedge \\ & \varphi \\ ). & \end{aligned}$$

By Lemma 10.2.2, any partially ordered set $\mathbb{J}(\downarrow x, \leq)$ for $x \in M$ has width at most m. Hence by Dilworth' Theorem, there are mutually disjoint chains C_i that cover P. Now the statement of the theorem follows by the consideration above.

□

Since the set of join-irreducible elements of a distributive lattice is definable inside the lattice, we obtain as a direct consequence of the theorem above the following

Corollary 11.1.7 *Let $(M, \cdot, 1)$ be a divisibility monoid with finitely many residuum functions. There exists a monadic sentence ψ such that for any finite partial order $(P, \leq)$ it holds:*

$$(P, \leq) \models \psi \iff \exists x \in M : (\downarrow x, \leq) \cong (P, \leq).$$

Let $(M, \cdot, 1)$ be a width-bounded divisibility monoid. To show that in this case the monadic theory $\mathrm{MTh}(\{(\downarrow x, \leq) \mid x \in M\})$ is decidable, we now show that the set of lattices $\{(\downarrow x, \leq) \mid x \in M\}$ is contained in a set of lattices whose monadic theory is decidable. Then, by the corollary above, the decidability of $\mathrm{MTh}(\{(\downarrow x, \leq) \mid x \in M\})$ follows easily:

An undirected graph (T, K) is a *tree* if for any $s, t \in T$, there is a unique path connecting s and t. Now let (V, E) be a finite directed graph and $n \in \mathbb{N}$. Then (V, E) has *tree-width at most n* if there exists a tree (T, K) and a mapping $\psi : T \to 2^V$ such that

1. for any $(x, y) \in E$, there is $t \in T$ with $x, y \in \psi(T)$,
2. for any $s, t, u \in T$ such that t is on the path connecting s and u, we have $\psi(s) \cap \psi(u) \subseteq \psi(t)$,
3. $\bigcup_{s \in T} \psi(i) = V$, and
4. $|\psi(t)| < n$ for any $t \in T$.

Lemma 11.1.8 *Let $n \in \mathbb{N}$ and $(L, \leq)$ a finite distributive lattice of width at most n. Then the graph $(L, -\!\!<)$ has tree width at most $2n$.*

Proof. Let m denote the length of L. The tree (T, K) that we construct is (the Hasse diagram of) the linear order on $\{1, 2, \dots, m\}$. Let $\psi(i)$ be the set of all vertices in $(L, \leq)$ of height $i-1$ or i.

Now let $x, y \in L$ with $x -\!\!< y$. Then, since L is distributive, $h(y) - h(x) = 1$, i.e. $x, y \in \psi(h(y))$. Hence the first property is satisfied. For the second note that $\psi(i) \cap \psi(k) = \emptyset$ whenever there is $i < j < k$. Hence it is trivially satisfied. Similarly, the third requirement holds trivially. Finally, $\psi(i)$ consists of two antichains. Since the size of these antichains is bounded by n, the last requirement $|\psi(i)| \leq 2n$ follows. □

Theorem 11.1.9 *Let $(M, \cdot, 1)$ be a divisibility monoid with finitely many residuum functions. Then the monadic theory $\mathrm{MTh}\{(\downarrow m, \leq) \mid m \in M\}$ is decidable iff M is width-bounded.*

Proof. First, let $(M, \cdot, 1)$ be width-bounded by n. Then any lattice $(\downarrow x, \leq)$ has tree-width at most $2n$ by the preceding lemma. Now let μ be a monadic sentence. Then, by Corollary 11.1.7, μ belongs to $\mathrm{MTh}(\{(\downarrow x, \leq) \mid x \in M\})$ iff $\psi \to \mu$ is satisfied by all finite distributive lattices of tree width at most $2n$. But this question is decidable by [Cou90].

If, on the other hand, $(M, \cdot, 1)$ is not width-bounded, by Theorem 10.2.6, any grid $([n]^2, \leq)$ can be embedded into some lattice $(\downarrow x, \leq)$. Since the monadic theory of these grids is undecidable, the monadic theory of all lattices $(\downarrow x, \leq)$ with $x \in M$ is undecidable. □

Let $(M, \cdot, 1)$ be a divisibility monoid and let $\mathfrak{L}$ denote the set of all distributive lattices $(\downarrow x, \leq)$ for $x \in M$. Then, by Theorem 11.1.3, the monadic theory of $\mathbb{J}(\mathfrak{L}) := \{\mathbb{J}(L, \leq) \mid (L, \leq) \in \mathfrak{L}\}$ is decidable. By the theorem above, $\mathrm{MTh}(\mathfrak{L})$ is decidable iff the width of the elements of $\mathfrak{L}$ is uniformly bounded. As an encore which is not directly related to divisibility monoids, we show in the following two sections that this last connection between the bounded width of a class of distributive lattices $\mathfrak{L}$ and the decidability of $\mathrm{MTh}(\mathbb{J}(\mathfrak{L}))$ holds in general and is not a particular feature of divisibility monoids.

11.2 The semilattice of finitely generated ideals

It is the aim of this section to relate the monadic theory of a set of partially ordered sets to the monadic theory of the semilattices of finitely generated ideals that are associated with these partially ordered sets. In particular, we are interested in the relation between the decidabilities of these theories.

Remark 11.2.1 Let $\mathfrak{P}$ be a set of partially ordered sets. Then $\mathrm{MTh}(\mathfrak{P})$ can be reduced in linear time to $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$.

Proof. Recall that a partially ordered set $(P, \leq)$ is isomorphic to $\mathbb{J}\,\mathbb{H}_f(P, \leq)$. Hence, a sentence is satisfied by $(P, \leq)$ iff its restriction to the join-irreducible elements is valid in $\mathbb{H}_f(P, \leq)$. Since this restriction can be computed in linear time, the statement follows. □

Theorem 11.2.2 *Let $\mathfrak{P}$ be a set of partially ordered sets and $n \in \mathbb{N}$ such that $w(P, \leq) \leq n$ for any $(P, \leq) \in \mathfrak{P}$. Then $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$ can be reduced to $\mathrm{Th}(\mathfrak{P})$ in linear time.*

Proof. The idea of the proof is that any finitely generated ideal in $(P, \leq)$, i.e. any element of $\mathbb{H}_f(P, \leq)$ is generated by at most n elements of P. Therefore, the reduction r is defined by

$$\begin{aligned} r(\exists x \alpha) &= (\exists x_1 \exists x_2 \ldots \exists x_n r(\alpha)), \\ r(x \leq y) &= (\bigwedge_{1 \leq i \leq n} \bigvee_{1 \leq j \leq n} x_i \leq y_j), \\ r(\alpha \vee \beta) &= (r(\alpha) \vee r(\beta)), \text{ and} \\ r(\neg \alpha) &= \neg r(\alpha). \end{aligned}$$

Identifying a tuple $(x_1, x_2, \ldots, x_n)$ in P with its ideal $x_1\!\downarrow \cup x_2\!\downarrow \cup \cdots \cup x_n\!\downarrow$, one easily verifies that

$$\mathbb{H}_f(P, \leq) \models \varphi \iff (P, \leq) \models r(\varphi)$$

for any elementary sentence φ and any $(P, \leq) \in \mathfrak{P}$. Hence, in particular, r reduces $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$ to $\mathrm{Th}(\mathfrak{P})$ in linear time. □

As an immediate consequence of the above, we obtain

Corollary 11.2.3 *Let $\mathfrak{P}$ be a set of partially ordered sets and $n \in \mathbb{N}$ such that $w(P, \leq) \leq n$ for any $(P, \leq) \in \mathfrak{P}$. Then* $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$ *is decidable iff* $\mathrm{Th}(\mathfrak{P})$ *is decidable.*

11.2.1 From $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$ to $\mathrm{MTh}(\mathfrak{P})$

Our next aim is to show a similar result for the monadic theory $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$. Note that the basic idea of the reduction of $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$ to $\mathrm{Th}(\mathfrak{P})$ is the replacement of an element of $\mathbb{H}_f(P, \leq)$ by a finite antichain in $(P, \leq)$. If we want to extend this to *sets* of elements in $\mathbb{H}_f(P, \leq)$, it would be natural to consider *sets of finite antichains* in $(P, \leq)$. But this is not possible in monadic second order logic. To make this possible at least in full second order logic, the size of antichains in $(P, \leq)$ should by bounded, i.e., the width of the elements of $\mathfrak{P}$ has to be bounded. Then, a set in $\mathbb{H}_f(P, \leq)$ could be represented by an n-ary relation in $(P, \leq)$. But in monadic second order logic, we cannot quantify over relations, but only over sets. Therefore, it does not suffice to restrict attention to sets $\mathfrak{P}$ of bounded width, but we have in addition to assume that the diabolo width of the elements of $\mathfrak{P}$ is bounded:

Definition 11.2.4 The partial order $(P, \leq)$ has *diabolo width at most m* if, for any $X, Y \subseteq P$ such that $X \times Y \subseteq \|$, we have $|X| \leq m$ or $|Y| \leq m$.

Figure 11.2 depicts this notion: Let $(P, \leq)$ be a partially ordered set of diabolo width at most m and let $X \subseteq P$ be a set with more than m elements. Then the set $Y := P \setminus (X\uparrow \cup X\downarrow)$ is incomparable with X. Hence it contains at most m elements.[1]

Note that the width is at most double the diabolo width of a partially ordered set.

In this section, we will show that for any set $\mathfrak{P}$ of partial orders of bounded diabolo width, the monadic theory of $\mathbb{H}_f(\mathfrak{P})$ can be reduced to the monadic theory of $\mathfrak{P}$. In particular, we have to ensure that any n-ary relation in $\mathfrak{P}$ that contains only antichains can be encoded by a bounded number of sets.

We start with two technical lemmas.

Lemma 11.2.5 *Let $(C, \leq)$ be a linearly ordered set and let $k \in \mathbb{N}$. Then C splits into $2k$ mutually disjoint subsets $C(j)$ $(1 \leq j \leq 2k)$ satisfying*

($\star$) *For any $1 \leq j \leq 2k$ and for any $x, y \in C(j)$ with $x < y$, the interval $x\uparrow \cap y\downarrow \subseteq C$ contains at least k elements.*

Proof. Let α be an ordinal and let $C = \{x_\beta \mid \beta < \alpha\}$ be an enumeration of C. By transfinite induction, we construct the subsets $C(j)$ as follows: Let $\beta < \alpha$ and

[1] The name "diabolo width" was chosen since in this picture the set $X\uparrow \cup X\downarrow$ looks like a diabolo – a juggling prop that the author hopes to master eventually.

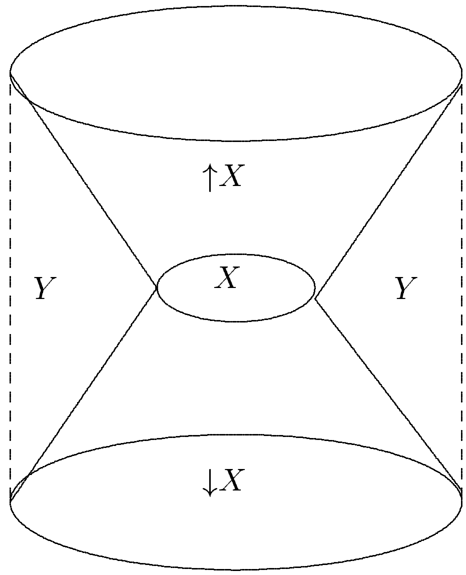

Figure 11.2: Diabolo width

assume that we constructed a partition $(C^\beta(j))_{1\leq j\leq 2k}$ of $\{x_\gamma \mid \gamma < \beta\}$ satisfying $(\star)$. Consider the sets

$$\begin{aligned} M &= \{x \in C \mid x < x_\beta, |x\uparrow \cap x_\beta\downarrow| \leq k\} \text{ and} \\ N &= \{x \in C \mid x > x_\beta, |x_\beta\uparrow \cap x\downarrow| \leq k\}. \end{aligned}$$

Since they are linearly ordered, M and N both contain at most $k-1$ elements. Hence there is $1 \leq j \leq 2k$ with $C^\beta(j) \cap (M \cup N) = \emptyset$. Now define $C^{\beta+1}(j) := C^\beta(j) \cup \{x_\beta\}$ and $C^{\beta+1}(i) := C^\beta(i)$ for $i \neq j$. Then $(C^{\beta+1}(j))_{1\leq j\leq 2k}$ is a partition of $\{x_\gamma \mid \gamma < \beta+1\}$ satisfying $(\star)$. For a limit ordinal β, we set $C^\beta(j) := \bigcup_{\gamma<\beta} C^\gamma(j)$ for $1 \leq j \leq 2k$. Now $C(j) := C^\alpha(j)$ finishes the construction. □

Let $(P, \leq)$ be a partial order and let $C_1, C_2 \subseteq P$ be chains in $(P, \leq)$. We define an equivalence relation $\sim$ on C_2 by $x \sim y$ iff $C_1 \cap x\updownarrow = C_1 \cap y\updownarrow$, i.e. iff x and y are comparable with the same elements of the first chain C_1.

Note that except $C_1 \cap C_2$ all equivalence classes of $\sim$ are intervals in C_2. But since $C_1 \cap C_2$ is an equivalence class, in the following lemma we might have $x_i \sim x_j$ for some $i < j$.

Lemma 11.2.6 *Let $(P, \leq)$ be a partial order of diabolo width at most m. Let $C_1, C_2 \subseteq P$ be chains in $(P, \leq)$ and let $k = (2m+3)^2$. Let $x_i \in C_2$ with $x_i \not\sim x_{i+1}$ and $x_i < x_{i+1}$ for $1 \leq i < k$. Then $C_1 \subseteq x_1\uparrow \cup x_k\downarrow$, i.e. there is no element in C_1 that is incomparable with both x_1 and x_k.*

Proof. For $1 \leq i \leq k$, let $\Downarrow x_i = \{x \in C_1 \mid x < x_i\}$ and $\Uparrow x_i = \{x \in C_1 \mid x > x_i\}$. Then $\Downarrow x_i \cup \Uparrow x_i$ consists of those elements from $C_1 \setminus \{x_i\}$ that are comparable with x_i. Hence $\Downarrow x_i \subseteq \Downarrow x_{i+1}$ and $\Uparrow x_i \supseteq \Uparrow x_{i+1}$. Since $x_i \not\sim x_{i+1}$ and $x_i < x_{i+1}$, we have $\Downarrow x_i \subsetneq \Downarrow x_{i+1}$ or $\Uparrow x_i \supsetneq \Uparrow x_{i+1}$ for $1 \leq i < k$. First we show $\Downarrow x_i \subsetneq \Downarrow x_{i+2m+3}$ for $1 \leq i \leq k - 2m - 3$:

By contradiction, assume $1 \leq i \leq k - 2m - 3$ with $\Downarrow x_i = \Downarrow x_{i+2m+3}$. Let $x \in X := \Uparrow x_i \setminus \Uparrow x_{i+m+1}$ and let $m + 1 < \ell \leq 2m + 3$. If x is comparable with $x_{i+\ell}$, we get $x < x_{i+\ell}$ for otherwise $x_{i+m+1} < x_{i+\ell} \leq x$. Thus $x \in \Downarrow x_{i+\ell} = \Downarrow x_i$, i.e. $x \in \Downarrow x_i \cap \Uparrow x_i$, a contradiction since this set is empty. Thus, any element of X is incomparable with $x_{i+\ell}$ for $m + 1 < \ell \leq 2m + 3$, i.e. X and $Y := \{x_{i+m+2}, x_{i+m+3} \ldots x_{i+2m+3}\}$ are incomparable sets. Since Y contains more than m elements and since the diabolo width of $(P, \leq)$ is m, X contains at most m elements implying that there is $0 \leq j < m$ with $\Uparrow x_{i+j} \setminus \Uparrow x_{i+j+1} = \emptyset$. But this contradicts our assumption that $x_{i+j} \not\sim x_{i+j+1}$. Thus, we proved $\Downarrow x_i \subsetneq \Downarrow x_{i+2m+3}$.

Now let $y_i := x_{(2m+3)i}$ for $1 \leq i \leq \ell := \frac{k}{2m+3} = 2m + 3$, i.e. $y_1 < y_2 < \cdots < y_\ell$ is a subsequence of $x_1 < x_2 < \cdots < x_k$ such that $\Downarrow y_i \subsetneq \Downarrow y_{i+1}$ for $1 \leq i < \ell$.

To prove the final goal $C_1 \subseteq \uparrow x_1 \cup \downarrow x_k$, assume by contradiction that x is an element of $C_1 \setminus (x_1 \uparrow \cup x_k \downarrow)$ and let $1 \leq i \leq m + 1$ and $z \in Z := \Downarrow y_{2m+3} \setminus \Downarrow y_{m+2}$. Since z and x belong to the chain C_1, they are comparable. Then $z < x$ for otherwise $x \leq z \leq y_{2m+3} \leq x_k$ would contradict our assumption on x. In case $z \geq y_i$ ($z \leq y_i$), we had $x > z \geq y_i \geq x_1$ ($z \leq y_i \leq y_{m+2}$), contradicting our assumption on x (on z, respectively). Hence z and y_i are incomparable, i.e. the sets $\{y_1, y_2, \ldots, y_{m+1}\}$ and Z are incomparable. Since they both contain more than m elements, the diabolo width of $(P, \leq)$ is larger than m, a contradiction.□

Let $(P, \leq)$ be a partially ordered set. By a slight abuse of notation, we call an n-tuple $(x_1, x_2, \ldots, x_n) \in P^n$ an *antichain* if the set $\{x_1, x_2, \ldots, x_n\}$ is an antichain. Note that in particular the tuple $(a, a, \ldots, a)$ is an antichain for any $a \in P$. By Antichains, we denote the set of all *tuples* that are antichains.

Recall that our aim was to encode an n-ary relation whose elements form antichains by subsets of P. With the following lemma, we reach this aim for the restricted case where $(P, \leq)$ has width at most 2.

Lemma 11.2.7 *Let $(P, \leq)$ be a partial order of diabolo width at most m. Let $C_1, C_2 \subseteq P$ be chains in $(P, \leq)$. For any $M \subseteq C_1 \times C_2 \cap$ Antichains, there exist sets $M_{i,j} \subseteq C_i$ for $i = 1, 2$ and $0 \leq j < 4m(2m + 3)^2$ such that*

$$M = \left(\bigcup_{j=0}^{4m(2m+3)^2} M_{1,j} \times M_{2,j} \right) \cap \text{Antichains}.$$

Proof. Similarly to the preceding lemma, let $k = (2m + 3)^2$. First, we split the chain C_2 into the set C_2^s of those elements that belong to a small $\sim$-equivalence class and its complement, i.e.

$$\begin{aligned} C_2^s &= \{y \in C_2 : |[y]| \leq m\}, \text{ and} \\ C_2^l &= C_2 \setminus C_2^s = \{y \in C_2 : |[y]| > m\} \end{aligned}$$

where $[y]$ denotes the $\sim$-equivalence class containing y. Using Lemma 11.2.5, next we split C_1 and C_2^s into $2{\cdot}k{\cdot}m$ disjoint subchains $C_1(j)$ and $C_2^s(j)$ for $1 \leq j \leq 2km$ such that, for any $x, y \in C_1(j)$ ($\in C_2^s(j)$, respectively) with $x < y$ the interval $\uparrow x \cap \downarrow y$ contains at least km elements from $C_1(j)$ (from $C_2^s(j)$, respectively). This ensures that between any two elements of $C_1(j)$, there are more than m elements of C_1. Similarly, we will use that between two elements of $C_2^s(j)$ there are at least $k \cdot m$ elements of C_2^s. To finish the construction, let

$$\begin{aligned} M^l(j) &= M \cap C_1(j) \times C_2^l, \text{ and} \\ M^s(j) &= M \cap C_1 \times C_2^s(j) \end{aligned}$$

for $1 \leq j \leq 2km$. We establish the lemma showing that M equals the set of antichains that occur in

$$H = \bigcup_{1 \leq j \leq 2km} \left[\pi_1(M^l(j)) \times \pi_2(M^l(j)) \cup \pi_1(M^s(j)) \times \pi_2(M^s(j))\right].$$

Let $(x, y) \in M$. In case $y \in C_2^l$, there is $1 \leq j \leq 2km$ with $x \in C_1(j)$. Hence $(x, y) \in M \cap C_1(j) \times C_2^l = M^l(j)$. Now $(x, y) \in \pi_1(M^l(j)) \times \pi_2(M^l(j)) \subseteq H$ follows immediately. In case $y \in C_2^s$, we find $1 \leq j \leq 2km$ with $y \in C_2^s(j)$. Now $(x, y) \in \pi_1(M^s(j)) \times \pi_2(M^s(j)) \subseteq H$ follows, i.e. we showed $M \subseteq H \cap \text{Antichains}$.

Conversely, we have to show that antichains from $\pi_1(M^l(j)) \times \pi_2(M^l(j))$ or from $\pi_1(M^s(j)) \times \pi_2(M^s(j))$ belong to M for any $1 \leq j \leq 2km$. So let $1 \leq j \leq 2km$ and $(x_1, x_2), (y_1, y_2) \in M^l(j)$ with $x_1 \parallel y_2$. We want to show $x_1 = y_1$ implying $(x_1, y_2) = (y_1, y_2) \in M$. By contradiction assume $x_1 \neq y_1$. Since $x_1, y_1 \in C_1(j)$, they are comparable. We assume $x_1 < y_1$ (the case $y_1 < x_1$ is dual). As remarked earlier, there are more than m elements of C_1 between x_1 and y_1, in particular $|x_1 \uparrow \cap y_1 \downarrow| > m$. All elements of this interval are incomparable with y_2 since its endpoints x_1 and y_1 are. Thus we found incomparable sets $x_1 \uparrow \cap y_1 \downarrow$ and $[y_2]$ both larger than m. Since this contradicts the assumption on the diabolo width of $(P, \leq)$, we obtain $x_1 = y_1$ and therefore

$$\pi_1(M^l(j)) \times \pi_2(M^l(j)) \cap \text{Antichains} \subseteq M \text{ for any } 1 \leq j \leq 2km.$$

Finally, let $1 \leq j \leq 2km$ and $(x_1, x_2), (y_1, y_2) \in M^s(j)$ with $x_1 \parallel y_2$. To show $x_2 = y_2$, we now assume by contradiction $x_2 < y_2$. Similarly to above, there are at least km elements of C_2^s in the interval $x_2 \uparrow \cap y_2 \downarrow$. Since $|[x]| \leq m$ for any $x \in C_2^s$, the chain $C_2^s \cap x_2 \uparrow \cap y_2 \downarrow$ contains k mutually not $\sim$-equivalent elements. Hence, by Lemma 11.2.6, $C_1 \setminus (x_2 \uparrow \cap y_2 \downarrow) = \emptyset$, contradicting $x_1 \in C_1$ and $x_2 \parallel x_1 \parallel y_2$.$\square$

Next, we extend the lemma above to relations of larger arity.

Theorem 11.2.8 *Let $(P, \leq)$ be a partial order of diabolo width at most m and let $n > 1$. Let $C_i \subseteq P$ be chains for $1 \leq i \leq n$ and let $M \subseteq \prod_{1\leq i\leq n} C_i \cap \text{Antichains}$. Then M is the intersection of* Antichains *with $(2m+2)^n$ sets of the form*

$$\bigcap_{1\leq a<b\leq n} \bigcup_{\ell=1}^{4m(2m+3)^2} \left[P^{a-1} \times M_\ell^{a,b} \times P^{b-a-2} \times N_\ell^{a,b} \times P^{n-b-1}\right] \qquad (\star)$$

where $M_\ell^{a,b}, N_\ell^{a,b} \subseteq P$ for all suitable a, b and ℓ.

Proof. By Lemma 11.2.5, we split the chains C_i into $2m+2$ disjoint subchains $C_i(j)$ for $1 \leq j \leq 2m$ such that for any $x, y \in C_i(j)$ with $x < y$ the interval $\uparrow x \cap \downarrow y$ contains at least $m+1$ elements from P. For $\vec{\jmath} \in \{1, 2, \ldots, 2m+2\}^n$ let $M_{\vec{\jmath}} = M \cap \prod_{1\leq i\leq n} C_i(j_i)$. Then M is the union of the sets $M_{\vec{\jmath}}$. Since there are $(2m+2)^n$ sets $M_{\vec{\jmath}}$, it suffices to show that any such set is the intersection of Antichains with a set of the form $(\star)$. Let $1 \leq a < b \leq n$. Applying Lemma 11.2.7 to the set $\pi_{a,b}(M_{\vec{\jmath}})$ and the chains $C_a(j_a)$ and $C_b(j_b)$, we obtain the existence of sets $M_\ell^{a,b} \subseteq C_a(j_a)$ and $N_\ell^{a,b} \subseteq C_b(j_b)$ for $1 \leq \ell \leq 4m(2m+3)^2$ such that

$$\pi_{a,b}(M_{\vec{\jmath}}) = \bigcup_{\ell=1}^{4m(2m+3)^2} M_\ell^{a,b} \times N_\ell^{a,b} \cap \text{Antichains}.$$

Let $H_{\vec{\jmath}}$ denote the set of all antichains in

$$\bigcap_{1\leq a<b\leq n} \bigcup_{\ell=1}^{4m(2m+3)^2} \left[P^{a-1} \times M_\ell^{a,b} \times P^{b-a-2} \times N_\ell^{a,b} \times P^{n-b-1}\right].$$

Note that $H_{\vec{\jmath}}$ equals the set of all antichains $(x_1, x_2, \ldots, x_n) \in P^n$ such that $(x_a, x_b) \in \pi_{a,b}(M_{\vec{\jmath}})$ for any $1 \leq a < b \leq n$. Thus, in particular, $H_{\vec{\jmath}}$ is a subset of $\prod_{1\leq i\leq n} C_i(j_i)$ since $\pi_{a,b}(M_{\vec{\jmath}}) \subseteq C_a(j_a) \times C_b(j_b)$. Since $H_{\vec{\jmath}}$ is of the form $(\star)$, it remains to show that $M_{\vec{\jmath}} = H_{\vec{\jmath}}$.

The inclusion $M_{\vec{\jmath}} \subseteq H_{\vec{\jmath}}$ is immediate for any element $(x_1, x_2, \ldots, x_n)$ of $M_{\vec{\jmath}}$ is an antichain satisfying $(x_a, x_b) \in \pi_{a,b}(M_{\vec{\jmath}})$ for all suitable a, b.

We show by induction on the size of $I \subseteq \{1, 2, \ldots, n\}$ that $\pi_I(H_{\vec{\jmath}}) \subseteq \pi_I(M_{\vec{\jmath}})$ which, for $I = \{1, 2, \ldots, n\}$ establishes the claim and therefore the theorem. If I contains precisely two elements, the inclusion $\pi_I(H_{\vec{\jmath}}) \subseteq \pi_I(M_{\vec{\jmath}})$ is immediate by what we said above. Now let I contain at least three elements and assume that $\pi_J(H_{\vec{\jmath}}) \subseteq \pi_J(M_{\vec{\jmath}})$ for any proper subset J of I. For notational simplicity, we assume $I = \{1, 2, \ldots, c\}$ for some $3 \leq c \leq n$. Let $(x_1, x_2, \ldots, x_c) \in \pi_I(H_{\vec{\jmath}})$. Then, by the induction hypothesis, there are elements $x_i^i \in C_i(j_i)$ such that

$$(x_1, \ldots, x_{i-1}, x_i^i, x_{i+1}, \ldots, x_c) \in \pi_I(M_{\vec{\jmath}})$$

for any $i \in I$. If for some $1 \leq i \leq c$ we even have $x_i = x_i^i$, we thus get immediately $(x_1, \ldots, x_c) \in \pi_I(M_{\vec{\jmath}})$. Now assume $x_i \neq x_i^i$ for all $1 \leq i \leq c$. Since

$x_i, x_i^i \in C_i(j_i)$, they are comparable. Since I contains at least three elements, there are $1 \le a < b \le c$ with $x_a < x_a^a$ and $x_b < x_b^b$ or with $x_a > x_a^a$ and $x_b > x_b^b$. By symmetry, it suffices to deal with the first case. Recall that $x_a \not\lessgtr x_b$ since $(x_1, \ldots, x_c)$ (as an element of $\pi_I(H_{\bar{\jmath}})$) is an antichain. Similarly, $x_a^a \not\lessgtr x_b$ and $x_b^b \not\lessgtr x_a$ since $(x_1, \ldots, x_{a-1}, x_a^a, x_{a+1}, \ldots, x_c)$ and $(x_1, \ldots, x_{b-1}, x_b^b, x_{b+1}, \ldots, x_c)$ are antichains as elements of $\pi_I(M_{\bar{\jmath}})$. Thus x_a^a and x_b^b are incomparable for otherwise $x_a^a \le x_b^b$ implied $x_a < x_b^b$. Since $\{x_a, x_a^a\}$ and $\{x_b, x_b^b\}$ are incomparable sets, so are the intervals $x_a{\uparrow} \cap x_a^a{\downarrow}$ and $x_b{\uparrow} \cap x_b^b{\downarrow}$. Recall that $x_a < x_a^a$ are both elements of $C_a(j_a)$. Thus, the interval $x_a{\uparrow} \cap x_a^a{\downarrow}$ consists of at least $m+1$ elements, and similarly the interval $x_b{\uparrow} \cap x_b^b{\downarrow}$. This contradicts the assumption that $(P, \le)$ has diabolo width at most m. Therefore, it is impossible that $x_i \ne x_i^i$ for all $1 \le i \le c$. This finishes the induction step, i.e. we have indeed $\pi_I(H_{\bar{\jmath}}) \subseteq \pi_I(M_{\bar{\jmath}})$. □

The following corollary shows that we can indeed represent any set of antichains in $(P, \le)$ by some subsets of P. The number of subsets necessary is effectively bounded by the diabolo width of $(P, \le)$.

Corollary 11.2.9 *Let $m \in \mathbb{N}$ and $n = 2m$. Then there exists a natural number ℓ and a monadic formula $\varphi(x_1, \ldots, x_n, X_1, X_2, \ldots, X_\ell)$ such that, for any partially ordered set $(P, \le)$ of diabolo width at most m and any set R of antichains in P, there are sets $M_1, \ldots, M_\ell \subseteq P$ with*

$$R = \{\{x_1, x_2, \ldots, x_n\} \subseteq P \mid (P, \le) \models \varphi(x_1, \ldots, x_n, M_1, \ldots M_\ell\}.$$

Proof. We explain the idea of the formula and leave the technicalities to the interested reader: We are concerned with partially ordered sets of width at most n, only. Hence any element of R contains at most n elements. By Dilworth' Theorem [Dil50], the partially ordered set $(P, \le)$ can be covered by n disjoint chains $C_1, \ldots, C_n$. For $I \subseteq \{1, 2, \ldots, n\}$, let R_I denote the set of all antichains in R that meet a chain C_i iff $i \in I$. In particular, the set R_I contains sets of size $|I|$, only. Thus, we can identify it with an $|I|$-ary relation on P such that $R_I \subseteq \prod_{i \in I} C_i$. Now, applying Theorem 11.2.8, we easily construct a formula φ_I with $|I|$ free elementary variables and $(2m+2)^{|I|} n(n-1) 4m(2m+3)^2 \cdot 2$ free set variables such that there exist sets M_i with

$$R_I = \{(y_1, \ldots, y_{|I|}) \mid (P, \le) \models \varphi_I(y_i, M_i)\}.$$

The formula φ is a simple Boolean combination of the formulas φ_I. □

Theorem 11.2.10 *Let $\mathfrak{P}$ be a set of partially ordered sets and $m \in \mathbb{N}$ such that any $(P, \le)$ in $\mathfrak{P}$ has diabolo width at most m. Then $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$ can be reduced to $\mathrm{MTh}(\mathfrak{P})$ in linear time.*

Proof. Let φ and ℓ denote the formula and the natural number from Corollary 11.2.9. The reduction r is defined by

$$\begin{array}{rcl} r(\exists x\alpha) & = & (\exists x_1 \exists x_2 \ldots \exists x_n r(\alpha)), \\ r(x \leq y) & = & (\bigwedge_{1\leq i\leq n} \bigvee_{1\leq j\leq n} x_i \leq y_j), \\ r(\exists M\alpha) & = & (\exists M_1 \exists M_2 \ldots \exists M_\ell r(\alpha)), \\ r(x \in M) & = & \varphi(x_1, x_2, \ldots, x_n, M_1, M_2, \ldots, M_\ell), \\ r(\alpha \vee \beta) & = & (r(\alpha) \vee r(\beta)), \text{ and} \\ r(\neg\alpha) & = & \neg r(\alpha). \end{array}$$

Differently from the proof of Theorem 11.2.2, we spell out the equivalence $\mathbb{H}_f(P, \leq) \models \varphi \iff (P, \leq) \models r(\varphi)$ in some more detail:

Let V be a countable set of individual variables and W that of set variables. We use these variables in monadic formulas that are interpreted over $\mathbb{H}_f(P, \leq)$. By $V' := V \times \{1, 2, \ldots, n\}$ and $W' := W \times \{1, 2, \ldots, \ell\}$ we denote the individual and set variables when speaking on the partial orders in $\mathfrak{P}$. For simplicity, we abbreviate (x, i) by x_i and similarly (A, j) by A_j for $x \in V$ and $A \in W$.

Let $(P, \leq)$ be a partially ordered set in $\mathfrak{P}$. With any $a \in \mathbb{H}_f(P, \leq)$, we associate an n-tuple $f'(a)$ in P with $\{f'(a)_1, f'(a)_2, \ldots, f'(a)_n\} = \max(a)$. Such an n-tuple exists since a is a finitely generated ideal in $(P, \leq)$ implying that it has at most n maximal elements. Furthermore, $a = \bigcup_{1\leq i\leq n} f'(a)_i{\downarrow}$. Note that the coordinates of the tuple $f'(a)$ are incomparable if not equal.

Similarly, we find a function g' that maps subsets of $\mathbb{H}_f(P, \leq)$ to ℓ-tuples of subsets of P as follows: Let $M \subseteq \mathbb{H}_f(P, \leq)$ be a set of finitely generated ideals in $(P, \leq)$. By R, we denote the set of all $(\leq n)$-subsets $\max(a)$ of P for some $a \in M$, i.e. $R = \{\max(a) \mid a \in M\}$. Then R is a set of antichains in the partially ordered set $(P, \leq)$ of diabolo width at most m. Hence, by Corollary 11.2.9 there exist sets $M_1, M_2, \ldots, M_\ell \subseteq P$ with

$$R = \{\{x_1, x_2, \ldots, x_n\} \subseteq P \mid (P, \leq) \models \varphi(x_1, \ldots, x_n, M_1, \ldots M_\ell\}.$$

For $1 \leq j \leq \ell$, let $g'(M)_j := M_j$. Then we obtain for any $M \subseteq \mathbb{H}_f(P, \leq)$ and any $a \in \mathbb{H}_f(P, \leq)$:

$$a \in M \text{ iff } (P, \leq) \models \varphi(f'(a), g'(M)).$$

Now let (f, g) be an interpretation of the elementary variables V and the set variables W in $\mathbb{H}_f(P, \leq)$, i.e. $f : V \to \mathbb{H}_f(P, \leq)$ and $g : W \to 2^{\mathbb{H}_f(P,\leq)}$. By $f^\star(x_i) := (f' \circ f(x))_i$ and $g^\star(M_j) := (g' \circ g(M))_j$ for $x_i \in V'$ and $M_j \in W'$, we define an interpretation $(f^\star, g^\star)$ of V' and W' in $(P, \leq)$ from (f, g).

To finish the proof, one shows by induction on the monadic formula φ that $\mathbb{H}_f(P, \leq) \models_{(f,g)} \varphi$ iff $(P, \leq) \models_{(f^\star,g^\star)} r(\varphi)$. This is an easy exercise which is left to the reader. □

Since undecidable theories cannot be reduced to decidable theories, we obtain as a direct consequence

Corollary 11.2.11 *Let $\mathfrak{P}$ be a set of partially ordered sets whose diabolo width is bounded such that the monadic theory* $\mathrm{MTh}(\mathfrak{P})$ *is decidable. Then the monadic theory* $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$ *is decidable as well.*

Recall that the elements of $(P, \leq)$ correspond to the join irreducible elements of $\mathbb{H}_f(P, \leq)$. Hence it is easily seen that the decidability of $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$ implies that of $\mathrm{MTh}(\mathfrak{P})$. To obtain the inverse implication of the corollary above, it remains to show that the monadic theory of $\mathbb{H}_f(\mathfrak{P})$ can be decidable only in case $\mathfrak{P}$ has bounded diabolo width.

11.2.2 Decidable monadic theory implies bounded diabolo width

By contraposition, we will actually show that the monadic theory of $\mathbb{H}_f(\mathfrak{P})$ is undecidable whenever the diabolo width of $\mathfrak{P}$ is unbounded. This will be achieved by reducing the monadic theory of the set of finite two-dimensional grids $\mathcal{G}$ to $\mathrm{MTh}(\mathfrak{P})$. We are even a bit more ambitious and will also consider the monadic antichain and monadic chain theory.

It will be convenient to use the notation $\underline{m}$ for the set $\{0, 1, \ldots, m-1\}$ of nonnegative integers properly smaller than m. Let $\mathcal{G}$ denote the set of all finite grids, seen as distributive lattices, i.e. the set of all partial orders $(\underline{m}, \leq) \times (\underline{n}, \leq)$ for $m, n > 1$. We will show that

(a) the monadic theory of $\mathcal{G}$ is undecidable,

(b) the monadic theory of $\mathcal{G}$ can be reduced to the monadic antichain theory of $\mathcal{G}$, and this monadic antichain theory can be reduced to the monadic chain theory of $\mathcal{G}$, and

(c) the monadic chain (antichain, resp.) theory of $\mathcal{G}$ can be reduced to the monadic chain (antichain, resp.) theory of $\mathbb{H}_f(\mathfrak{P})$.

Given a Turing machine M, one can effectively formulate a monadic sentence μ which is satisfied by the partially ordered set $L = (\underline{m} \times \underline{n}, \leq)$ iff the machine stops after m steps using n cells of the tape. Hence $\mu \in \mathrm{MTh}(\mathcal{G})$ iff M does not stop, i.e. the monadic theory of $\mathcal{G}$ is indeed undecidable which establishes (a). For this encoding, one has to quantify over arbitrary subsets of the grid. Therefore, the undecidability of the monadic chain and antichain theory of $\mathcal{G}$ is not immediate.

Figure 11.3 indicates an outline of the remainder of this section. In this figure, arrows denote reductions.

We start with the reductions in the first row, i.e., we first consider the different monadic theories of the set of grids.

$$\begin{array}{ccccc} \mathrm{MTh}(\mathcal{G}) & \xrightarrow{\text{Lemma 11.2.13}} & \mathrm{MATh}(\mathcal{G}) & \xrightarrow{\text{Lemma 11.2.14}} & \mathrm{MCTh}(\mathcal{G}) \\ & & \big\downarrow \text{Lemma 11.2.16} & & \text{Lemma 11.2.18} \big\downarrow \\ & & \mathrm{MATh}(\mathbb{H}_f(\mathfrak{P})) & & \mathrm{MCTh}(\mathbb{H}_f(\mathfrak{P})) \end{array}$$

Figure 11.3: Outline of Section 11.2.2

The set of grids

The *grid graph of dimension* (m, n) is the Hasse diagram $(\underline{m} \times \underline{n}, E_{m,n})$ of the grid $(\underline{m} \times \underline{n}, \leq)$. Hence $((i, j), (i', j')) \in E_{m,n}$ iff $i = i'$ and $j + 1 = j'$ or $i + 1 = i$ and $j = j'$ for any $(i, j), (i', j') \in \underline{m} \times \underline{n}$. Let $\mathcal{GG}$ denote the set of all structures isomorphic to some grid graph. We first show how a grid graph can be encoded by antichains in a large grid:

Lemma 11.2.12 *Let $m, n \in \mathbb{N}$. Then there exist antichains X, E_1, E_2 and an element e in the grid $L = (\underline{(mn + 1)}^2, \leq)$ such that*

$$(X, \{(x, y) \in X^2 \mid \sup(x, y) \in E_1 \cup E_2 \wedge \inf(x, e) \leq \inf(y, e)\} \cong (\underline{m} \times \underline{n}, E_{m,n}).$$

Proof. First, we define the following sets:
$X := \{(i, mn - i) \mid 0 \leq i < mn\}$,
$E_1 := \{(i, mn - i - 1) \mid 0 \leq i < mn, i \mod n \neq n - 1\}$, and
$E_2 := \{(i, (m - 1)n - i) \mid 0 \leq i < (m - 1)n\}$. Note that these sets are antichains for increasing i increases the first and decreases the second component of any of their elements. Finally, let $e := (mn, 1) \in \mathbb{J}(L)$. For notational simplicity, let E denote the binary relation

$$\{(x, y) \in X^2 \mid \sup(x, y) \in E_1 \cup E_2 \wedge \inf(x, e) \leq \inf(y, e)\}.$$

We show that (X, E) and $(\underline{m} \times \underline{n}, E_{m,n})$ are isomorphic: Define the bijection $f : \underline{m} \times \underline{n} \to X$ by $f(a_1, a_2) := (a_1 n + a_2, mn - a_1 n - a_2)$. The following sequence of equivalences establishes that f is a graph isomorphism:

$$((a_1, a_2), (b_1, b_2)) \in E_{m,n}$$

$\iff$ $a_1 + 1 = b_1$ and $a_2 = b_2$, or
$a_1 = b_1$ and $a_2 + 1 = b_2$

$\iff$ $a_1 \leq b_1$ and either
$(a_1 n + a_2, mn - b_1 n - b_2) \in E_1$, or
$(a_1 n + a_2, mn - b_1 n - b_2) \in E_2$

$\iff$ $\inf(f(a_1, a_2), e) = a_1 \leq b_1 = \inf(f(b_1, b_2), e)$ and $\sup(f(a_1, a_2), f(b_1, b_2)) \in E_1 \cup E_2$

$\iff$ $(f(a_1, a_2), f(b_1, b_2)) \in E$.

Hence $(X, E) \cong (\underline{m} \times \underline{n}, E_{m,n})$. □

Lemma 11.2.13 *The monadic theory of $\mathcal{G}$ can be reduced to the monadic antichain theory of $\mathcal{G}$ in linear time.*

Proof. Since the grid graph of dimension (m, n) is the Hasse diagram of the grid $(\underline{m} \times \underline{n}, \leq)$, it suffices to reduce the monadic theory of the set of all grid graphs to the monadic antichain theory of $\mathcal{G}$.

For a monadic sentence φ over the binary relation symbol E, we construct a monadic formula over the vocabulary $\leq$ as follows: First, we restrict the quantification in φ to the new set variable X. Afterwards, any subformula of the form $(x, y) \in E$ is replaced by

$$\sup(x, y) \in E_1 \cup E_2 \wedge \inf(x, e) \leq \inf(y, e).$$

Let φ' denote the result of this procedure. Then φ' is a monadic formula with free variables contained in $\{X, E_1, E_2, e\}$.

Now we describe the reduction of $\mathrm{MTh}(\mathcal{GG})$ to $\mathrm{MATh}(\mathcal{G})$: It is easily seen that there is a monadic sentence γ such that a graph (X, E) satisfies γ iff it belongs to $\mathcal{GG}$. Let φ be a monadic sentence over the binary relation symbol E. Then define

$$\overline{\varphi} := \forall X \forall E_1 \forall E_2 \forall e((e \in \mathbb{J} \wedge \gamma') \rightarrow \varphi').$$

We show that φ belongs to $\mathrm{MTh}(\mathcal{GG})$ iff $\overline{\varphi}$ belongs to $\mathrm{MATh}(\mathcal{G})$:

First let $\varphi \in \mathrm{MTh}(\mathcal{GG})$. Furthermore, let $m, n > 1$ and $L := (\underline{m} \times \underline{n}, \leq)$. Let $X, E_1, E_2 \subseteq L$ be antichains and $e \in \mathbb{J}(L)$ with $(L, \leq) \models_A \gamma'(X, E_1, E_2, e)$. We have to show that $(L, \leq) \models \varphi'(X, E_1, E_2, e)$. First, define a binary relation $E \subseteq X^2$ by $(x, y) \in E$ iff $\sup(x, y) \in E_1 \cup E_2$ and $\inf(x, e) \leq \inf(y, e)$. By the construction of γ' and the fact that $(L, \leq) \models_A \gamma'(X, E_1, E_2, e)$, the graph (X, E) satisfies γ, i.e. it is isomorphic to a grid graph. Hence $(X, E) \models \varphi$ implying $(L, \leq) \models_A \varphi'(X, E_1, E_2, e)$ as required.

Conversely, let $\overline{\varphi} \in \mathrm{MATh}(\mathcal{G})$ and let $m, n \in \mathbb{N}$. To show $(\underline{m} \times \underline{n}, E_{n,m}) \models \varphi$, we consider the grid $L = (\underline{(mn+1)}^2, \leq)$. By Lemma 11.2.12, there exist antichains X, E_1, E_2 and a join irreducible element e such that (X, E) is isomorphic to $(\underline{m} \times \underline{n}, E_{n,m})$ (where E is defined as above). Hence $(X, E) \models \gamma$. By the construction of E and of γ', this implies $L \models_A \gamma'(X, E_1, E_2, e)$ and therefore $L \models_A \varphi'(X, E_1, E_2, e)$. By the same argument, $(X, E) \models \varphi$. □

Next, we reduce the monadic antichain theory of $\mathcal{G}$ to the monadic chain theory of $\mathcal{G}$. The basic idea is depicted in Figure 11.4: It shows the set $\underline{3} \times \underline{4}$. On this set, we consider two orders: first the natural order $\leq$, i.e., the componentwise order. The second order $\sqsubseteq$ in consideration is obtained by inverting the order on $\underline{4}$ and then considering the componentwise order. Then elements incomparable w.r.t. $\sqsubseteq$ are comparable w.r.t. $\leq$ which is the basis for our transformation of antichains into chains. Unfortunately, the inverse does not hold (consider, e.g., $(0,0)$ and $(1,0)$). Hence, we do not have $(\underline{m} \times \underline{n}, \leq) \models_C \varphi$ iff $(\underline{m} \times \underline{n}, \sqsubseteq) \models_A \varphi$. Therefore, we have to invest slightly more work to reduce the monadic antichain theory to the monadic chain theory.

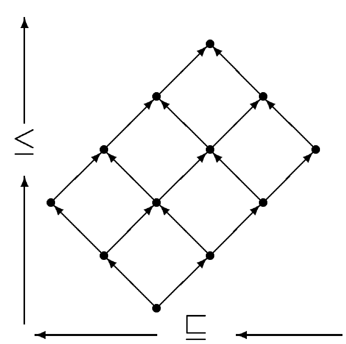

Figure 11.4: The set $\underline{3} \times \underline{4}$

More formally, our reduction rests on the following definition: Let $n, m > 1$ and $L := (\underline{m} \times \underline{n}, \leq)$. Then $e := (1, m)$ and $\overline{e} := (n, 1)$ are maximal join irreducible and incomparable elements of L. We define a partial order $\sqsubseteq$ on L by $x \sqsubseteq y$ iff $\inf(x, e) \leq \inf(y, e)$ and $\inf(x, \overline{e}) \geq \inf(y, \overline{e})$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, it holds $\inf(x, e) = x_2$ and $\inf(x, \overline{e}) = x_1$. Hence $x \sqsubseteq y$ iff $x_2 \leq y_2$ and $x_1 \geq y_1$. In other words, $(\underline{m} \times \underline{n}, \sqsubseteq)$ equals $(\underline{m}, \geq) \times (\underline{n}, \leq)$ and is therefore isomorphic to L. Now let x and y be incomparable with respect to $\sqsubseteq$. Then $x_1 < y_1$ and $x_2 < y_2$ or vice versa. In particular $x \leq y$ or $y \leq x$. Hence antichains in $(\underline{m} \times \underline{n}, \sqsubseteq)$ are chains in $(\underline{m} \times \underline{n}, \leq)$ (the converse implication does not hold).

Lemma 11.2.14 *The monadic antichain theory* MATh$(\mathcal{G})$ *can be reduced in linear time to the monadic chain theory* MCTh$(\mathcal{G})$.

Proof. Let φ be a monadic formula not containing the variables e and $\overline{e}$. In φ, replace any atomic formula $x \leq y$ by

$$\inf(x, e) \leq \inf(y, e) \wedge \inf(x, \overline{e}) \geq \inf(y, \overline{e})$$

and replace any subformula of the form $\exists X\psi$ by $\exists X(\text{antichain}_{\sqsubseteq}(X) \wedge \psi)$ where $\text{antichain}_{\sqsubseteq}(X)$ denotes the formula

$$\forall x, y((x, y \in X \wedge \inf(x, e) \leq \inf(y, e)) \to \inf(x, \overline{e}) \leq \inf(y, \overline{e})).$$

The subformula $\text{antichain}_{\sqsubseteq}(X)$ is satisfied by a set X iff its elements are mutually incomparable with respect to $\sqsubseteq$. Denote the result of these replacements by φ'.

Let $m, n \geq 1$, $e = (1, m)$ and $\overline{e} = (n, 1)$. As a prerequisite, we show that $(\underline{m} \times \underline{n}, \sqsubseteq) \models_A \varphi$ iff $(\underline{m} \times \underline{n}, \leq) \models_C \varphi'$ by induction: Clearly, $(\underline{m} \times \underline{n}, \sqsubseteq) \models_A x \leq y$ iff $x \sqsubseteq y$ iff $(\underline{m} \times \underline{n}, \leq) \models_C \inf(x, e) \leq \inf(y, e) \wedge \inf(x, \overline{e}) \geq \inf(y, \overline{e})$ which equals $(x \leq y)'$. Now let φ_i $(i = 1, 2)$ be monadic formulas such that for any antichains X_i w.r.t. $\sqsubseteq$ and elements x_j:

$$(\underline{m} \times \underline{n}, \sqsubseteq) \models_A \varphi_i(X_1, \ldots, X_k, x_1, \ldots, x_\ell)$$
$$\Longleftrightarrow$$
$$(\underline{m} \times \underline{n}, \leq) \models_C \varphi_i'(X_1, \ldots, X_k, x_1, \ldots, x_\ell).$$

It is straightforward to check that this equivalence holds for $\neg\varphi_1$, $\varphi_1 \wedge \varphi_2$ and for $\exists x_\ell \varphi_1$, too. The only nontrivial case in the induction is the formula $\varphi = \exists X_k \varphi_1$: So let $X_i \subseteq \underline{m} \times \underline{n}$ be antichains w.r.t. $\sqsubseteq$ for $1 \leq i < k$ and let $x_j \in \underline{m} \times \underline{n}$. Then $(\underline{m} \times \underline{n}, \sqsubseteq) \models_A \varphi(X_1, \ldots, X_{k-1}, x_1, \ldots, x_\ell)$ iff there exists an antichain X_k w.r.t. $\sqsubseteq$ such that $(\underline{m} \times \underline{n}, \sqsubseteq) \models_A \varphi_1(X_1, \ldots, X_k, x_1, \ldots, x_\ell)$. By the induction hypothesis, this is equivalent to $(\underline{m} \times \underline{n}, \leq) \models_C \varphi_1'(X_1, \ldots, X_k, x_1, \ldots, x_\ell)$. By the remarks preceding this lemma, X_k is a chain w.r.t. $\leq$. Hence the last statement is equivalent to $(\underline{m} \times \underline{n}, \leq) \models_C (\exists X_k(\text{antichain}_{\sqsubseteq}(X_k) \wedge \varphi_1'))(X_1, \ldots, X_{k-1}, x_1, \ldots, x_\ell)$ which equals φ'.

Now it is straightforward to show that for a monadic sentence φ it holds $(\underline{m} \times \underline{n}, \leq) \models_A \varphi$ iff $(\underline{m} \times \underline{n}, \leq) \models_C \exists e, \overline{e}(e, \overline{e} \in \max(\mathbb{J}) \wedge e \neq \overline{e} \wedge \varphi')$ (where $\mathbb{J}$ denotes the set of join irreducible elements) which is the desired reduction. $\square$

So far, we showed that the monadic theory of $\mathcal{G}$ can be reduced to its monadic chain and antichain theory. Hence these two theories are undeciable.

Sets of unbounded diabolo width

Next, we show the reductions from $\text{MATh}(\mathcal{G})$ and $\text{MCTh}(\mathcal{G})$ to $\text{MATh}(\mathbb{H}_f(\mathfrak{P}))$ and $\text{MCTh}(\mathbb{H}_f(\mathfrak{P}))$, resp., for $\mathfrak{P}$ a set of partial orders of unbounded diabolo width. Let $(P, \leq)$ be a finite partial order of diabolo width exceeding m. Then, by Lemma 10.2.5, there exists a lattice embedding of $\underline{m} \times \underline{m}$ into $(\mathbb{H}_f(P), \subseteq)$. The following lemma shows that such a grid can be defined in monadic antichain logic:

Lemma 11.2.15 *There exists a monadic formula $\gamma(A,b,x)$ such that for any partially ordered set $(P,\leq)$ the following hold:*

1. *for any antichain $A \subseteq \mathbb{H}_f(P,\leq)$ and any $b \in A$, the set $\{x \in \mathbb{H}_f(P,\leq) \mid \mathbb{H}_f(P,\leq) \models_A \gamma(A,b,x)\}$ is a grid, and*

2. *for any $n \in \mathbb{N}$ and any semilattice-embedding $f : (\underline{2n+1})^2 \to \mathbb{H}_f(P,\leq)$, there exist an antichain $A \subseteq \mathbb{H}_f(P,\leq)$ and an element $b \in A$ such that $\{x \in \mathbb{H}_f(P,\leq) \mid \mathbb{H}_f(P,\leq) \models_A \gamma(A,b,x)\} = \{f(n+i,n+j) \mid i,j \in \underline{n}\}$.*

Proof. Let $(P,\leq)$ be a partially ordered set, $A \subseteq P$ an antichain and $a \in A$. On A, we define a binary relation R_a by $(x,y) \in R_a$ iff $x \vee a \leq y \vee a$. There is a monadic formula φ with free variables A and a such that $(P,\leq) \models_A \varphi(A,a)$ iff

(A) (A,R_a) is a finite linear order, and

(B) for any $x,y,x',y' \in A$ with $(x,y),(x',y') \in R_a$ we have

$$x \vee y \leq x' \vee y' \iff (x,x'),(y',y) \in R_a.$$

Now let γ be the formula

$$\gamma(A,b,x) = b = x \vee \exists a \in A(\varphi(A,a) \wedge \exists a_1,a_2 \in A(b \leq x = \sup(a_1,a_2))).$$

We show the first statement: Let $G = \{x \in \mathbb{H}_f(P,\leq) \mid \mathbb{H}_f(P,\leq) \models_A \gamma(A,b,x)\}$. If there is no $a \in A$ satisfying φ, $G = \{b\}$ which is clearly a grid. So assume $a \in A$ satisfies φ. Let $\{a_0,a_1,\ldots,a_k\}$ be the enumeration of A with $(a_i,a_{i+1}) \in R_a$. Furthermore, let $b = a_m$. Now define $f(i,j) = \sup(a_{m-1},a_{m+j})$ for $(i,j) \in \underline{m+1} \times \underline{k-m+1}$. We show that this function an order isomorphism from $(\underline{m+1} \times \underline{k-m+1}, \leq)$ onto G: Let $(i,j),(i',j') \in \underline{m+1} \times \underline{k-m+1}$. Then $(i,j) \leq (i',j')$ iff $a_{m-i'} R_a a_{m-i}$ and $a_{m+j} R_a a_{m+j'}$. Since $m-i \leq m+j$, we have in addition $a_{m-i} R_a a_{m+j}$. Hence we can apply (B) and obtain that the last statement is equivalent to $\sup(a_{m-i},a_{m+j}) \leq \sup(a_{m-i'},a_{m+j'})$, i.e., to $f(i,j) \leq f(i',j')$. Thus, f is an order embedding into $\mathbb{H}_f(P,\leq)$. Next, we show that its image is G: For $(i,j) \in \underline{m+1} \times \underline{k-m+1}$, we have $a_{m-i} R_a b = a_m R_a a_{m+j}$ implying (by (B)) $b \leq \sup(a_{m-i},a_{m+j}) = f(i,j)$ which therefore belongs to G. Conversely, let $x = \sup(a_s,a_t) \in G$. Then (B) implies $s \leq m \leq t$ ensuring $x = f(m-s,t-m)$. Thus, f is indeed an order isomorphism onto G, i.e., G is a grid.

Next, we prove the second statement: Let $A = \{f(i,2n-i) \mid i \in \underline{2n}\}$, $a = f(0,2n)$, and $b = f(n,n)$. Then $f(i,2n) = \sup(a,f(i,2n-i)) \leq \sup(a,f(j,2n-j)) = f(j,2n)$ iff $i \leq j$. Hence the finite antichain $(A,\leq)$ is linearly ordered by the relation R_a, i.e., (A) holds. Next, let $i,j,i',j' \in \underline{2n}$ with $i \leq j$ and $i' \leq j'$. Then $\sup(f(i,2n-i),f(j,2n-j)) \leq \sup(f(i',2n-i'),f(j',2n-j'))$ iff $\max(i,j) \leq \max(i',j')$ and $\min(i,j) \geq \min(i',j')$. Since $i \leq j$ and $i' \leq j'$, this is equivalent to $i' \leq i$ and $j \leq j'$, i.e., we showed (B). Since $\varphi(A, a)$ is satisfied, $\sup(a_1, a_2) = x$ satisfies $\gamma(A, b, x)$ iff $x \geq b$ for $a_1, a_2 \in A$. Since $b = f(n, n)$, this is equivalent to $x \in \{f(n+i, n+j) \mid i, j \in \underline{n}\}$. □

This lemma enables us to show that indeed the monadic antichain theory of the grids $\mathcal{G}$ can be reduced to the monadic antichain theory of $\mathbb{H}_f(\mathfrak{P})$ whenever $\mathfrak{P}$ has unbounded diabolo width.

Lemma 11.2.16 *Let $\mathfrak{P}$ be a set of partially ordered sets such that the diabolo width of its members is not bounded above. Then the monadic antichain theory* $\mathrm{MATh}(\mathcal{G})$ *can be reduced to the monadic antichain theory* $\mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$.

Proof. For a monadic sentence ψ, let $r(\psi)$ denote the sentence $\forall A \forall b (b \in A \rightarrow \psi')$ where ψ' is the restriction of ψ to those elements x that satatisfy $\gamma(A, b, x)$ (cf. Lemma 11.2.15). By the preceding lemma, these elements form a grid, i.e., $\psi \in \mathrm{MATh}(\mathcal{G})$ implies $r(\psi) \in \mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$. Conversselly, let $r(\psi) \in \mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$ and let $n \in \mathbb{N}$. Then there exists $(P, \leq) \in \mathfrak{P}$ such that the grid $(\underline{2n^2}, \leq)$ can be embedded into $\mathbb{H}_f(P, \leq)$. Hence $\mathbb{H}_f(P, \leq)$ contains an antichain A and an element b such that $\{x \in \mathbb{H}_f(P, \leq) \mid \mathbb{H}_f(P, \leq) \models \gamma(A, b, x)\}$ is isomorphic to $(\underline{n}^2, \leq)$. Since $r(\psi) \in \mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$, we get $\mathbb{H}_f(P, \leq) \models_A \psi'$ and therefore $\psi \in \mathrm{MATh}(\mathcal{G})$. □

To show that also the monadic chain theory of the set of grids can be reduced to the monadic chain theory of $\mathbb{H}_f(\mathfrak{P})$, we proceed similarly to above: First, it is shown that large grids can be defined in the monadic chain logic, and then we prove that this yields the desired reduction:

Lemma 11.2.17 *There exists a monadic formula $\gamma(C_1, C_2)$ such that for any partially ordered set $(P, \leq)$ the following hold:*

1. *for any chains $C_1, C_2 \subseteq \mathbb{H}_f(P, \leq)$ such that $\mathbb{H}_f(P, \leq) \models_C \gamma(C_1, C_2)$, the set $\{\sup(x, y) \mid x \in C_1, y \in C_2\}$ is a grid, and*
2. *for any grid $G \subseteq \mathbb{H}_f(P, le)$, there exist chains $C_1, C_2 \in \mathbb{H}_f(P, \leq)$ such that $\mathbb{H}_f(P, \leq) \models_C \gamma(C_1, C_2)$ and the set $\{\sup(x, y) \mid x \in C_1, y \in C_2\}$ is isomorphic to G.*

Proof. The formula γ asserts that C_1 and C_2 are finite chains, $\min(C_1) = \min(C_2)$, the elements from $C_1 \setminus \{\min(C_1)\}$ are incomparable with the elements from $C_2 \setminus \{\min(C_2)\}$, and $\sup(x_1, x_2) \leq \sup(y_1, y_2)$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$ for any $x_i, y_i \in C_i$.

Let $f : (\underline{m} \times \underline{n}, \leq) \to \mathbb{H}_f(P, \leq)$ be a semilattice embedding. Then define $C_1 = \{f(i,0) \mid 0 \leq i < m\}$ and $C_2 = \{f(0,i) \mid) \leq i < n\}$. Then C_1 and C_2 are finite chains, $\min(C_1) = \min(C_2)$, and the elements from $C_1 \setminus \{\min(C_1)\}$ are incomparable with the elements from $C_2 \setminus \{\min(C_2)\}$. To see that $\sup(x_1, x_2) \leq \sup(y_1, y_2)$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$, note that $\sup(f(i_1, 0), f(0, i_2)) = f(i_1, i_2)$. Hence $\sup(f(i_1, 0), f(0, i_2)) \leq \sup(f(j_1, 0), f(0, j_2))$ iff $i_1 \leq j_1$ and $i_2 \leq j_2$, i.e., iff $f(i_1, 0) \leq f(j_1, 0)$ and $f(0, i_2) \leq f(0, j_2)$. Hence the chains C_1 and C_2 satisfy γ. Furthermore, the set $\{\sup(x, y) \mid x \in C_1, y \in C_2\}$ is the image of f and therefore isomorphic to the grid of dimension (m, n). This proves the second statement.

To prove the first statement, let $C_1 = \{f(0,0), f(1,0), \ldots, f(m,0)\})$ and $C_2 = \{f(0,0), f(0,1) \ldots, f(0,n)\}$ be finite chains satisfying γ. We can assume $f(0,i) \leq f(0, i+1)$ and $f(j, 0) \leq f(j+1, 0)$ for all suitable i and j. We show that $f : \underline{m} \times \underline{n} \to \mathbb{H}_f(P, \leq)$ defined by $f(i,j) = \sup(f(i,0), f(0,j))$ is a semilattice embedding; for this it actually suffices to show that it is an order embedding: Note that $f(i,j) \leq f(i',j')$ iff $f(i,0) \leq f(i',0)$ and $f(0,j) \leq f(0,j')$ by the last requirement expressed by γ. Hence, indeed, f is an order embedding. □

Lemma 11.2.18 *Let $\mathfrak{P}$ be a set of partially ordered sets such that the diabolo width of its members is not bounded above. Then the monadic chain theory* $\mathrm{MCTh}(\mathcal{G})$ *can be reduced to the monadic chain theory* $\mathrm{MCTh}(\mathbb{H}_f(\mathfrak{P}))$.

Proof. The reduction is given by $r(\psi) = \forall C_1, C_2(\gamma(C_1, C_2) \to \psi')$ where ψ' is the reduction of ψ to the set of suprema of elements of C_1 and C_2. The proof now proceeds similarly to the proof of Lemma 11.2.16. □

Recall that Theorem 11.2.2 characterized those classes of partial orders $\mathfrak{P}$ for which $\mathbb{H}_f(\mathfrak{P})$ has a decidable elementary theory. Now we can extend this to the monadic, the monadic chain, and the monadic antichain theory:

Theorem 11.2.19 *Let $\mathfrak{P}$ be a set of partially ordered sets.*

1. *The following are equivalent:*
   - *(i) The monadic theory* $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$ *is decidable.*
   - *(ii) The monadic chain theory* $\mathrm{MCTh}(\mathbb{H}_f(\mathfrak{P}))$ *is decidable.*
   - *(iii) the monadic theory* $\mathrm{MTh}(\mathfrak{P})$ *is decidable and the diabolo width of the elements of $\mathfrak{P}$ is bounded above.*
   - *(iv) the monadic chain theory* $\mathrm{MCTh}(\mathfrak{P})$ *is decidable and the diabolo width of the elements of $\mathfrak{P}$ is bounded above.*

2. *The monadic antichain theory* $\mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$ *is decidable if and only if the elementary theory* $\mathrm{Th}(\mathfrak{P})$ *is decidable and the diabolo width of the elements of* $\mathfrak{P}$ *is bounded above.*

Proof. The implication (i)⇒(ii) is trivial. Now assume $\mathrm{MCTh}(\mathbb{H}_f(\mathfrak{P}))$ to be decidable. Next we show the implication (ii)⇒(iv): The monadic chain theory $\mathrm{MCTh}(\mathfrak{P})$ is decidable since it can be interpreted in the monadic chain theory of $\mathbb{H}_f(\mathfrak{P})$. By contradiction, assume that the diabolo width of the elements of $\mathfrak{P}$ is unbounded. By Lemmas 11.2.13, 11.2.14, and 11.2.18 (cf. Figure 11.3), the monadic theory of the grids $\mathrm{MTh}(\mathcal{G})$ can be reduced to the monadic chain theory of $\mathbb{H}_f(\mathfrak{P})$, contradicting the decidability of this latter theory.

For the implication $(iv) \Rightarrow (iii)$ note that the width of the elements of $\mathfrak{P}$ is bounded by n, say. Hence any subset of P with $(P, \leq) \in \mathfrak{P}$ is the union n chains. Therefore, the monadic theory of $\mathfrak{P}$ can be reduced to the monadic chain theory of $\mathfrak{P}$. Hence (iii) holds. The last implication $(iii) \Rightarrow (i)$ follows from Theorem 11.2.10.

It remains to show the second statement: Suppose $\mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$ is decidable. Then, trivially, $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$ and therefore $\mathrm{Th}(\mathfrak{P})$ are decidable. The diabolo with of the elements of $\mathfrak{P}$ is bounded above by Lemmas 11.2.13 and 11.2.16. Thus, we showed one implication.

Conversely, let the diabolo width of the elements of $\mathfrak{P}$ be bounded by n and let $\mathrm{Th}(\mathfrak{P})$ be decidable. Since then the width of the elements of $\mathfrak{P}$ is bounded by $2n$, the elementary theory $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$ is decidable by Corollary 11.2.3. By Lemma 10.2.5, the width of the elements of $\mathbb{H}_f(\mathfrak{P})$ is bounded above by some $m \in \mathbb{N}$. Hence any antichain contains at most m elements implying that $\mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$ can be reduced to $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$. Hence the monadic antichain theory $\mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$ is decidable. □

11.3 Finite distributive lattices

Since, for any finite distributive lattice $(L, \leq)$ it holds $(L, \leq) \cong \mathbb{H}_f\, \mathbb{J}(L, \leq)$, we can now characterize the sets of finite distributive lattices having a decidable monadic (chain, antichain) theory:

Corollary 11.3.1 *Let* $\mathfrak{L}$ *be a set of finite distributive lattices.*

1. *The following are equivalent:*

   (i) The monadic theory $\mathrm{MTh}(\mathfrak{L})$ *is decidable.*

   (ii) The monadic chain theory $\mathrm{MCTh}(\mathfrak{L})$ *is decidable.*

(iii) *the monadic theory* $\mathrm{MTh}(\mathbb{J}(\mathfrak{L}))$ *is decidable and the width of the elements of* $\mathfrak{L}$ *is bounded above.*

(iv) *the monadic chain theory* $\mathrm{MCTh}(\mathbb{J}(\mathfrak{L}))$ *is decidable and the width of the elements of* $\mathfrak{L}$ *is bounded above.*

2. *The monadic antichain theory* $\mathrm{MATh}(\mathfrak{L})$ *is decidable if and only if the elementary theory* $\mathrm{Th}(\mathbb{J}(\mathfrak{L}))$ *is decidable and the width of the elements of* $\mathfrak{L}$ *is bounded above.*

Proof. Since $\mathbb{H}_f(\mathbb{J}(\mathfrak{L})) = \mathfrak{L}$, it remains to show that the width of the elements of $\mathfrak{L}$ is bounded if and only if the diabolo width of the elements of $\mathbb{J}(\mathfrak{L})$ is bounded. In the proof of Lemma 11.2.18 we saw that a bounded width of the elements of $\mathfrak{L}$ implies a bound of the diabolo width of the elements of $\mathbb{J}(\mathfrak{L})$.

To show the other implication assume $dw(\mathbb{J}(L, \leq)) < n - 2$ for any lattice $(L, \leq) \in \mathfrak{L}$. By contradiction, suppose that the width of the elements of $\mathfrak{L}$ is unbounded. Then there exists $(L, \leq)$ in $\mathfrak{L}$ such that $w(L, \leq) \geq R_{n+1}(6^n)$. By Theorem 10.2.6, there exists a lattice embedding $\eta : [n-1] \times [n-1] \to L$. Let $A := \mathbb{J}(L) \cap \downarrow\eta(1, n-1)$ and $B := \mathbb{J}(L) \cap \downarrow\eta(n-1, 1)$. Since the elements $\eta(1, i)$ and $\eta(j, 1)$ are pairwise incomparable for $i, j > 1$, $A \setminus B$ and $B \setminus A$ both contain at least $n - 2$ elements. Furthermore, these two sets are incomparable. Hence the diabolo width of $\mathbb{J}(L, \leq)$ is at least $n - 2$, a contradiction. □

Now let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be sets of finite distributive lattices. Suppose that the elementary theories of $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are decidable. Then, as an easy consequence of the Feferman-Vaught Theorem [FV59], the set of direct products of lattices from $\mathfrak{L}_1$ and lattices from $\mathfrak{L}_2$ has a decidable elementary theory. Next, we want to characterize when this set has a decidable monadic (chain, antichain) theory:

Corollary 11.3.2 *Let* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *be sets of finite distributive lattices and define* $\mathfrak{L} := \{(L_1, \leq) \times (L_2, \leq) \mid (L_i, \leq) \in \mathfrak{L}_i\}$. *If* $\mathrm{MTh}(\mathfrak{L}_i)$ *(*$\mathrm{MATh}(\mathfrak{L}_i)$, $\mathrm{MCTh}(\mathfrak{L}_i)$, *resp.) is decidable, then* $\mathrm{MTh}(\mathfrak{L})$ *(*$\mathrm{MATh}(\mathfrak{L})$, $\mathrm{MCTh}(\mathfrak{L})$, *resp.) is decidable iff* $\mathfrak{L}_1$ *or* $\mathfrak{L}_2$ *is finite.*

Proof. We give the proof for the monadic theories, only. The other cases can be handled similarly. If both $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are infinite, we find for any $n \in \mathbb{N}$ lattices $(L_1, \leq) \in \mathfrak{L}_1$ and $(L_2, \leq) \in \mathfrak{L}_2$ of length at least n. Then the width of the direct product $(L_1, \leq) \times (L_2, \leq)$ is at least n, i.e. the width of the lattices in $\mathfrak{L}$ is not bounded. Hence $\mathrm{MTh}(\mathfrak{L})$ is undecidable.

Conversely let $\mathfrak{L}_1$ be finite. Then there is $n \in \mathbb{N}$ with $|L_1| \leq n$ for any lattice $(L_1, \leq) \in \mathfrak{L}_1$. Since $\mathrm{MTh}(\mathfrak{L}_2)$ is decidable, we can assume $w(\mathfrak{L}_2) \leq n$. Note that the width $w(L_1 \times L_2, \leq)$ is at most $|L_1| \cdot w(L_2, \leq)$ for any finite distributive lattices $(L_1, \leq)$ and $(L_2, \leq)$. Hence $w(\mathfrak{L}) \leq n^2$. It remains to show that $\mathbb{J}(\mathfrak{L})$ has a decidable monadic theory: For finite distributive lattices $(L_1, \leq)$ and $(L_2, \leq)$, one has $\mathbb{J}(L_1 \times L_2, \leq) = \mathbb{J}(L_1, \leq) \dot{\cup} \mathbb{J}(L_2, \leq)$. Thus, we have to show that the monadic theory of $\{(P_1, \leq)\dot{\cup}(P_2, \leq) \mid (P_i, \leq) \in \mathbb{J}(\mathfrak{L}_i)\}$ is decidable. This follows from the composition theorem from Shelah [She75] (cf. [Tho97a] for the proof of this result) since $\mathrm{MTh}(\mathbb{J}(\mathfrak{L}_i))$ is decidable. □

Note that the in the corollary above we assumed from the very beginning that $\mathrm{MTh}(\mathfrak{L}_i)$ is decidable for $i = 1, 2$. Actually, the finiteness of $\mathfrak{L}_1$ or $\mathfrak{L}_2$ follows without this assumption from the decidability of $\mathrm{MTh}(\mathfrak{L})$. We finish this section with an example of classes $\mathfrak{L}_1$, $\mathfrak{L}_2$ and $\mathfrak{L}$ as in the corollary above such that $\mathfrak{L}_1$ is finite, $\mathfrak{L}$ has a decidable monadic theory but the monadic theory of $\mathfrak{L}_2$ is undecidable:

Example 11.3.3 For simplicity, let $\underline{2}$ denote the Boolean lattice $(\{1, 2\}, \leq)$. Let $\mathfrak{L}_1$ consist of the lattices $\underline{2}^i$ for $0 \leq i \leq 2$ (i.e. $\mathfrak{L}_1$ contains the one-point-lattice, the Boolean lattice and the diamond). Let Lin denote the set of finite linear orders and let $\mathfrak{P} \subseteq \mathrm{Lin}$ be an undecidable set of linear orders. We define a set of finite distributive lattices $\mathfrak{L}_2 \subseteq \{\underline{2}^i \times (L, \leq) \mid 0 \leq i \leq 2, (L, \leq) \in \mathrm{Lin}\}$ by $\underline{2}^i \times (L, \leq) \in \mathfrak{L}_2$ iff

1. $i \in \{0, 2\}$ and $(L, \leq) \in \mathrm{Lin}$, or
2. $i = 1$ and $(L, \leq) \in \mathfrak{P}$.

Then $\mathfrak{L}_2$ is a set of finite distributive lattices. It is undecidable since the subset of $\mathfrak{L}_2$ of lattices of width 2 corresponds to $\mathfrak{P}$ which was chosen to be undecidable. Hence, in particular, $\mathfrak{L}_2$ has an undecidable monadic theory. It is straightforward to show that the set of direct products of lattices from $\mathfrak{L}_1$ and $\mathfrak{L}_2$ equals the set $\{\underline{2}^i \times (L, \leq) \mid 1 \leq i \leq 5, (L, \leq) \in \mathrm{Lin}\} = \mathfrak{L}$. Since Lin has a decidable monadic theory, we can apply the corollary above and obtain that $\mathrm{MTh}(\mathfrak{L})$ is decidable.

Remark The important property that we used in this section is the isomorphism of $(L, \leq)$ and $\mathbb{H}_f\,\mathbb{J}(L, \leq)$ whenever $(L, \leq)$ is a finite distributive lattice. The reader may check that Corollary 11.3.1 holds verbatim if we require $\mathfrak{L}$ to consist of distributive lattices satisfying this isomorphism. In Corollary 11.3.2, we obtained that $\mathrm{MTh}(\mathfrak{L})$ is decidable if and only if $\mathfrak{L}_1$ or $\mathfrak{L}_2$ is a finite set of *finite* lattices.