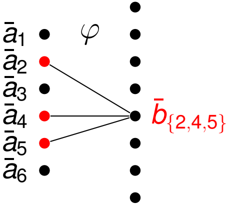
The VC dimension of definable sets in graph classes

Isolde Adler
Goethe University Frankfurt



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Outline

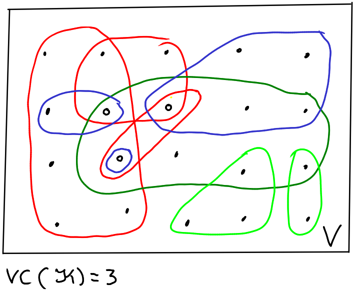
1. VC dimension
2. VC dimension of definable sets
3. Nowhere dense graph classes
4. Stability & bounded VC dimension
5. Conclusion

VC dimension

Definition

- For a set V, we call $\mathcal{K} \subseteq 2^V$ a **concept class.**
- For $U \subseteq V$ let $\mathcal{K} \upharpoonright U := \{X \cap U \mid X \in \mathcal{K}\}$.
  U is **shattered** by $\mathcal{K}$, if $\mathcal{K} \upharpoonright U = 2^U$.
- The **Vapnik-Chervonenkis (VC) dimension** of $\mathcal{K}$ is

$$\mathrm{VC}(\mathcal{K}) := \begin{cases} \max\{|U| \mid U \subseteq V \text{ shattered by } \mathcal{K}\}, & \text{if max exists,} \\ \infty, & \text{otherwise.} \end{cases}$$



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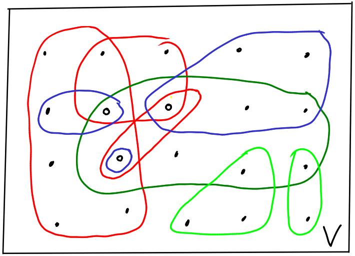

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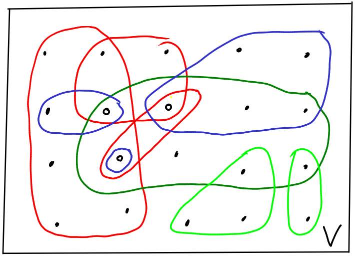

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PAC learning

- Successful learning of an unknown target concept $X \in \mathcal{K}$: Obtain with high probability a hypothesis $H \in \mathcal{K}$ that is a good approximation of X. PAC: Probably Approximately Correct.
- How to obtain H?
  Draw random examples $e \in V$ labeled '$+$' if $e \in X$ and '$-$' otherwise, and produce a consistent hypothesis.

Definition

Let $0 < \varepsilon, \delta < 1$. ε: error, $1 - \delta$: confidence.

$\mathcal{K}$ is **PAC-learnable with sample size** $s = s(1/\varepsilon, 1/\delta)$, if:

$\exists$ algorithm that, given ε and δ, draws s random examples of an unknown target concept $X \in \mathcal{K}$ according to distribution D on V, produces a hypothesis $H \in \mathcal{K}$ such that

$$\Pr_D(\Pr_D(X \Delta H) < \varepsilon) > 1 - \delta.$$

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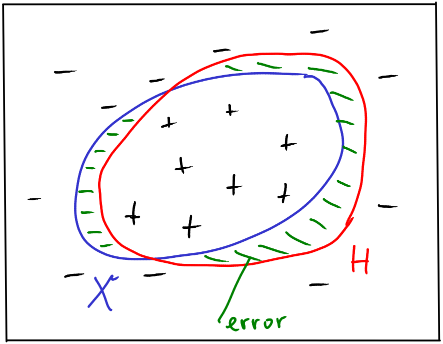

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Theorem (Blumer et al., Vapnik and Cervonenkis)

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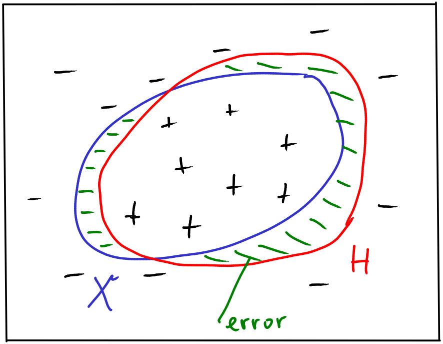

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Preliminaries

- we consider first-order logic (FO) and monadic second-order logic (MSO)
- MSO $=$ FO $+$ quantificaton over subsets of the universe
- relational structures, mostly undirected graphs
- Free variables are always individual variables

Definition

- For formula $\varphi(\bar{x}, \bar{y})$, structure M and elements $\bar{b}$ of M let

$$\varphi(M, \bar{b}) := \{\bar{a} \in M \mid M \models \varphi(\bar{a}, \bar{b})\}$$

  be the set **defined** by φ in M with parameters $\bar{b}$.
- Let $\mathcal{K}(\varphi, M) := \{\varphi(M, \bar{b}) \mid \bar{b} \in M\}$ be the concept class of all φ-definable subsets of M.

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Graphs are ...

undirected and simple.

$V(G) :=$ vertex set of G
$E(G) :=$ set of edges $e \subseteq V(G)$, $|e| = 2$

We encode undirected graphs as $\{E\}$-structures, where E is a symmetric, irreflexive binary relation

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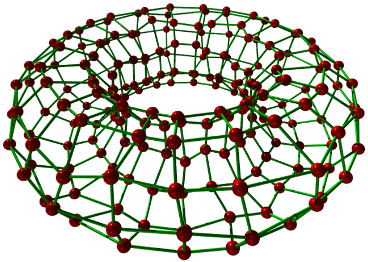

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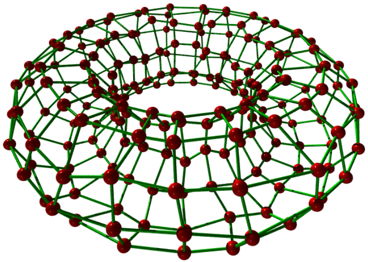

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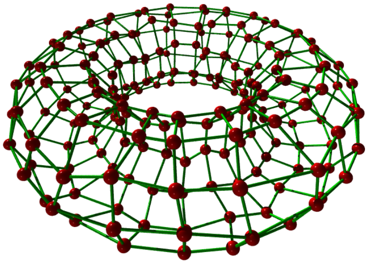

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Formulas of bounded VC dimension

Let $\mathcal{L} \in \{\text{FO}, \text{MSO}\}$. Let $\mathcal{C}$ be a class of structures of a fixed signature.

Definition

A formula $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}$ has **bounded VC dimension** on $\mathcal{C}$, if there is a d such that for every $M \in \mathcal{C}$ we have $\text{VC}\big((\mathcal{K}(\varphi, M)\big) \leq d$.

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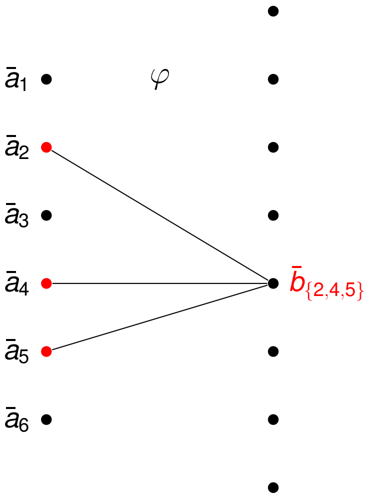
Formula φ has unbounded VC dimension



$$M \models \varphi(\bar{a}_i, \bar{b}_J) \iff i \in J$$

Example

Example (Grohe, Turán 2004)

MSO has unbounded VC dimension on the class of all square grid graphs $\{G_{n\times n} \mid n \in \mathbb{N}\}$.

- Define $\varphi(x; y_1, y_2) \in$ MSO such that for all $n \geq 1$: $\mathrm{VC}\big(\mathcal{K}(\varphi, G_{n\times n})\big) \geq \log(n)$.
- For $i \in \{1, \ldots, n\}$ formula φ satisfies: $(0, j) \in \varphi(G_{n\times n}, (0,0), (i,0)) \iff$ the jth bit in the binary representation of i is 1.
- Then: $\mathcal{K}(\varphi, G_{n\times n})$ shatters $\{(0, j) \mid 0 \leq j < \log(n)\}$

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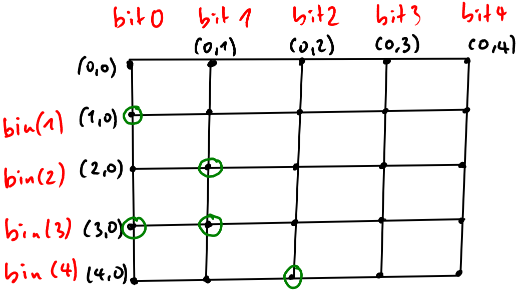
φ says:

1. $\exists$ set X such that all $p, q \in \{0, \ldots, n\}$ satisfy
   $(p, q) \in X \iff$ the qth bit of the binary representation of p is 1.

   For this, we say that the $(p + 1)$st row is one plus the pth row (for $p \in \{1, \ldots, n - 1\}$), where we read the rows as binary numbers with the elements of X being the ones, starting with the least significant bit.



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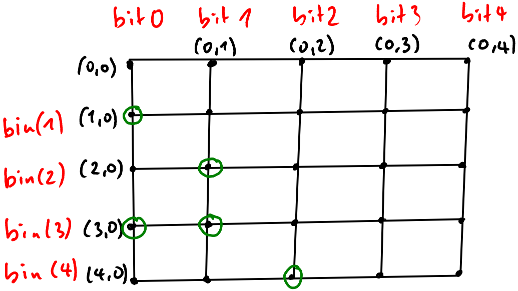

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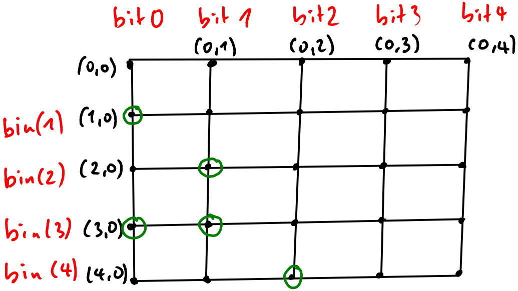

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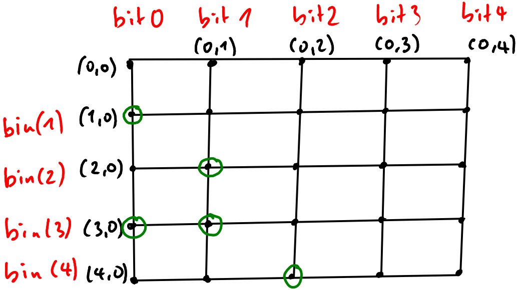

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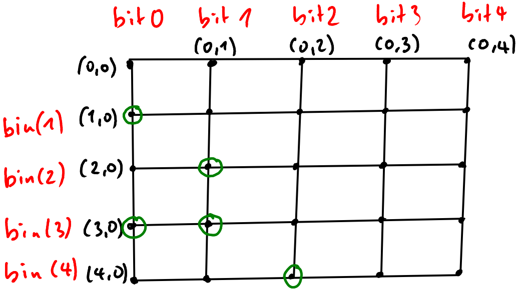

Main theorems

Theorem (Grohe, Turán 2004)

For any graph class $\mathcal{C}$ that is closed under taking subgraphs, the following are equivalent:

1. MSO *has bounded VC dimension on* $\mathcal{C}$
2. $\mathcal{C}$ *has bounded treewidth*

Theorem (Adler, Adler 2010)

For any graph class $\mathcal{C}$ that is closed under taking subgraphs, the following are equivalent:

1. FO *has bounded VC dimension on* $\mathcal{C}$
2. $\mathcal{C}$ *is nowhere dense*

Grohe-Turán Theorem: proof sketch

Theorem (Grohe, Turán 2004)

For any graph class $\mathcal{C}$ that is closed under taking subgraphs, the following are equivalent:

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Proof.

- 2⇒1: Use: If $\mathcal{C}'$ is a class of labeled binary trees, then MSO has bounded VC dimension on $\mathcal{C}'$. Encode graphs of bounded treewidth in labeled binary trees.
- 1⇒2: If $\mathcal{C}$ has unbounded treewidth, then $\mathcal{C}$ contains arbitrarily large square 'grids' as subgraphs. By the previous example: MSO has unbounded VC dimension on $\mathcal{C}$.

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Adler^2 Theorem

We show:

Theorem (A^2 2010)

For any graph class $\mathcal{C}$ that is closed under taking subgraphs, the following are equivalent:

- *$\mathcal{C}$ is nowhere dense,*
- *$\mathcal{C}$ is stable,*
- FO *has bounded VC dimension on $\mathcal{C}$.*

Topological minors

Definition

A **subdivision** of a graph H is a graph resulting from H by **subdividing** edges, i.e. by replacing edges by (new) paths.

H is a **topological minor** of G if a subdivision of H is isomorphic to a subgraph of G.

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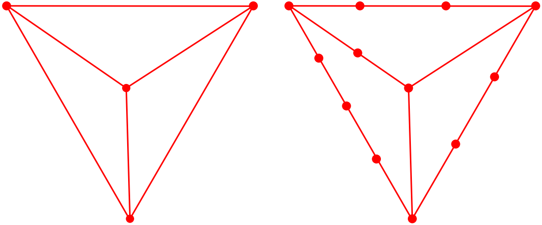

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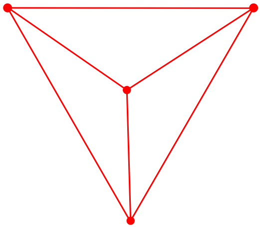

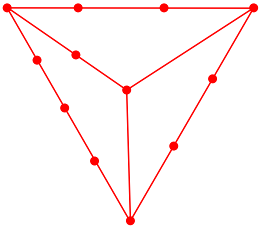

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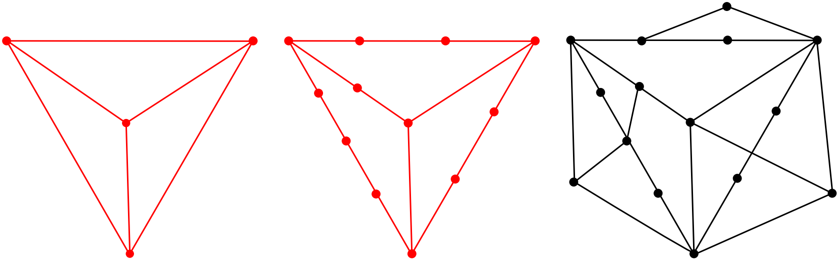

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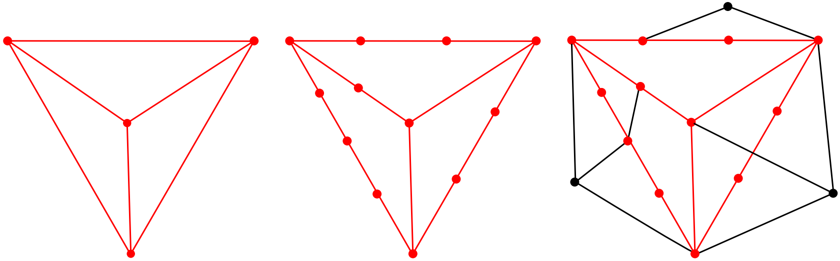

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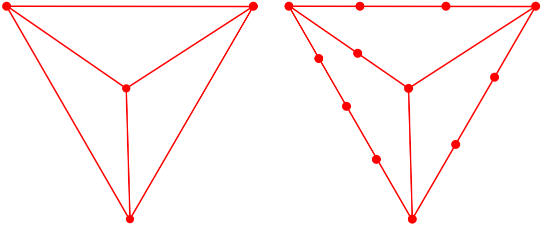

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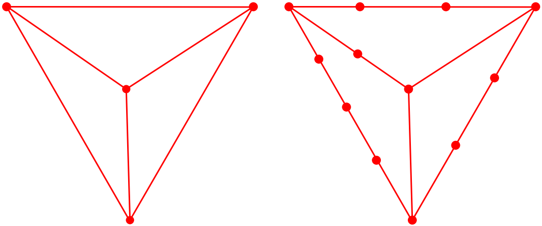

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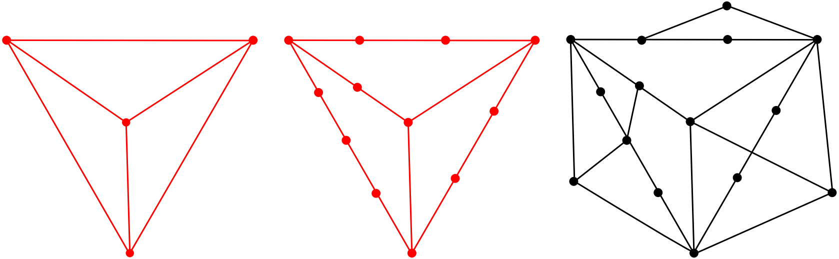

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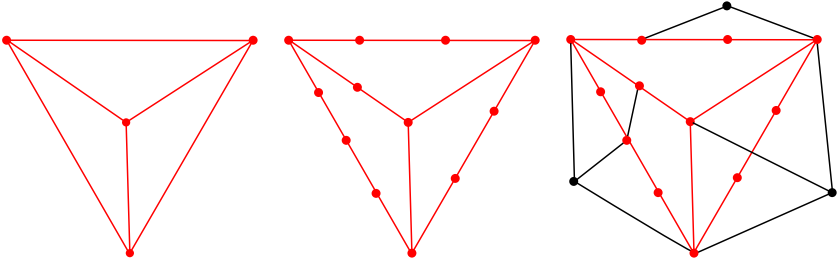

Nowhere dense graph classes

$K_n :=$ complete graph (clique) on n vertices

Definition (Nešetřil and de Mendez in 2008[1])

Let $\mathcal{C}$ be a graph class.
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Examples

- every finite graph class
- acyclic graphs $(r \mapsto n := 3)$
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Nowhere dense = superflat

Let $K_n^r := K_n$, where every edge is subdivided exactly r times.

Definition (Podewski and Ziegler, 1978)

Class $\mathcal{C}$ is **superflat**, if for every $r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that K_n^r is not a subgraph of any member of $\mathcal{C}$.

Example

- The class $\{K_r^r \mid 2 \leq r \in \mathbb{N}\}$ is superflat.

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$\mathcal{C}$ is superflat $\iff$ $\mathcal{C}$ is nowhere dense.

Proof: Ramsey. □

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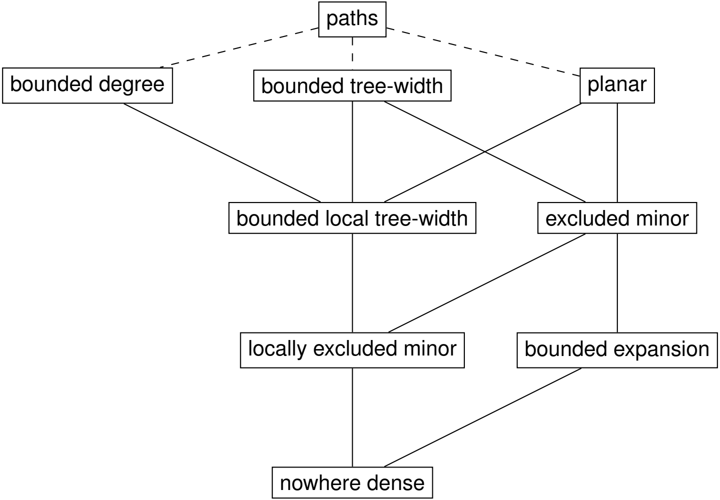

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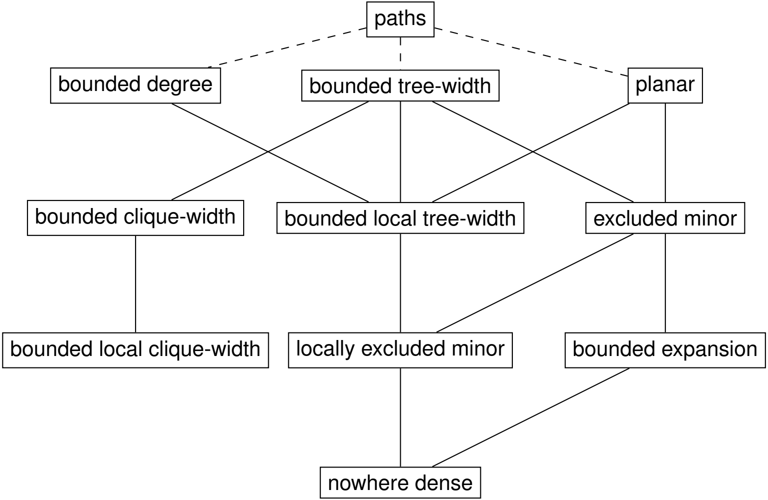

Outline

1. VC dimension
2. VC dimension of definable sets
3. Nowhere dense graph classes
4. Stability & bounded VC dimension
5. Conclusion

Adler^2 Theorem

We show:

Theorem (A^2 2010)

For any graph class $\mathcal{C}$ that is closed under taking subgraphs, the following are equivalent:

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Stability

Let $\mathcal{C}$ be a class of structures of a fixed signature.

Definition

A first-order formula $\varphi(\bar{x}, \bar{y})$ has the **order property** on $\mathcal{C}$ if for every n there exist a structure $M \in \mathcal{C}$ and tuples $\bar{a}_1, \ldots, \bar{a}_n, \bar{b}_1, \ldots, \bar{b}_n \in M$ such that

$$M \models \varphi(\bar{a}_i, \bar{b}_j) \iff i < j.$$

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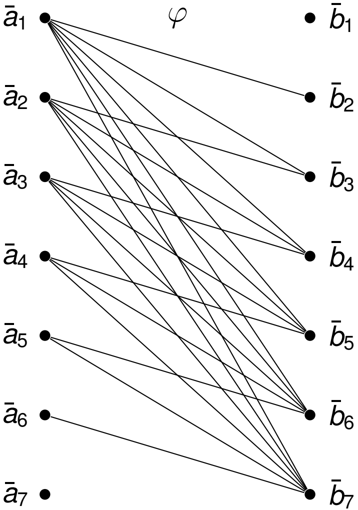

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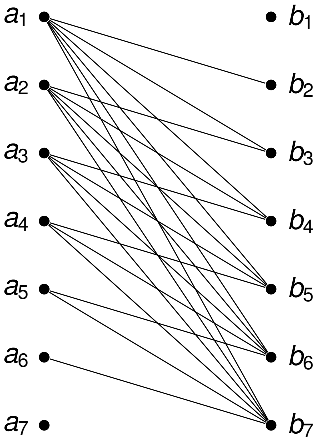
Example

The class of graphs B_n (where B_7 is shown below) is not stable, witnessed by $E(x, y)$.



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Podewski-Ziegler for graph classes

Theorem (Podewski, Ziegler 1978)

Let G be an infinite graph (coded as an $\{E\}$-structure). If $\{G\}$ is superflat then $\{G\}$ is stable. □

Lemma (Podewski-Ziegler for graph classes)

Let $\mathcal{C}$ be a graph class. If $\mathcal{C}$ is superflat then $\mathcal{C}$ is stable.

Proof sketch.

- Encode $\mathcal{C}$ in a single graph $G_\mathcal{C}$ s.t. $\mathcal{C}$ superflat $\Rightarrow \{G_\mathcal{C}\}$ superflat
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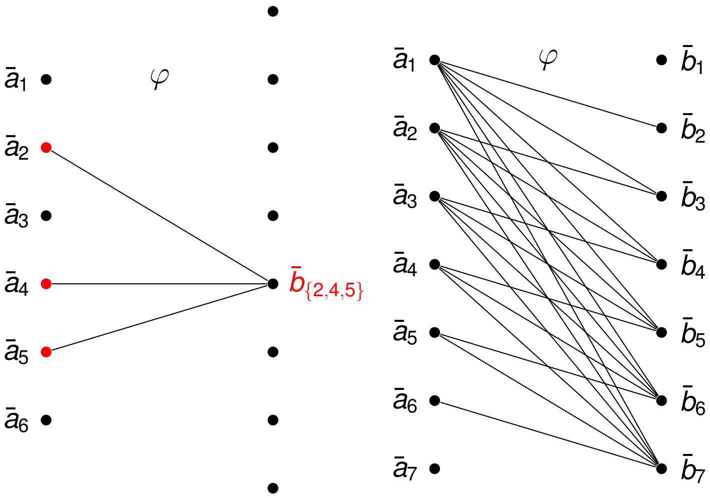
Stability & FO *has bounded VC dimension*

Remark

If $\mathcal{C}$ is stable then FO *has bounded VC dimension on $\mathcal{C}$.*

FO *unbounded VC dimension on* $\mathcal{C} \Rightarrow \mathcal{C}$ *not stable*



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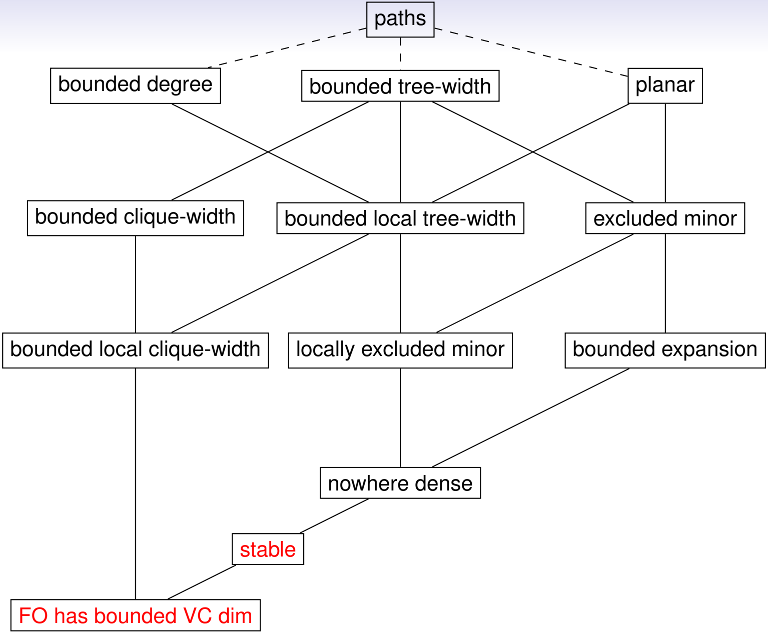
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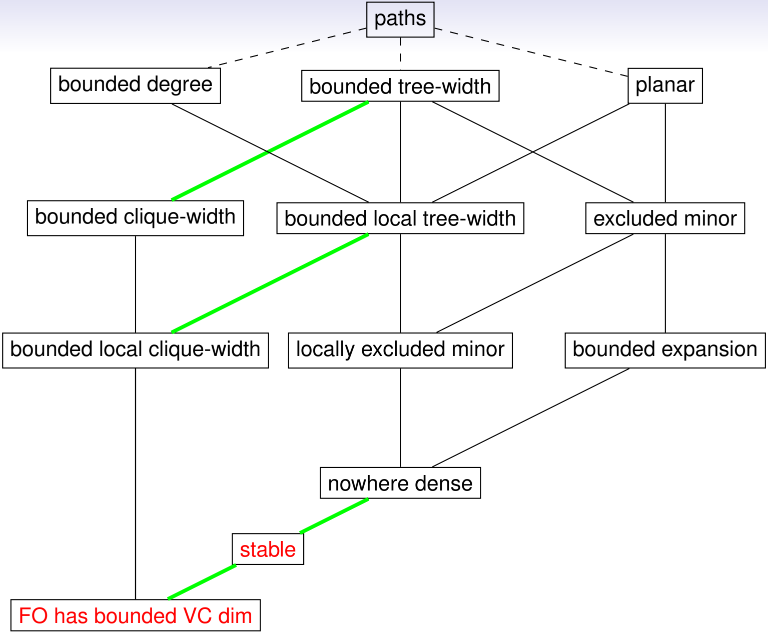
locally excluded minor
bounded expansion
nowhere dense
stable
FO has bounded VC dim


paths
bounded degree
bounded tree-width
planar
bounded clique-width
bounded local tree-width
excluded minor
bounded local clique-width
locally excluded minor
bounded expansion
nowhere dense
stable
FO has bounded VC dim

Outline

Conclusion: Outlook

Theorem (A² 2010)

$\mathcal{C}$ *a class of structures over a fixed finite signature of arity* ≤ 2,
$\underline{\mathcal{C}}$ *closed under subgraphs. The following are equivalent.*

1. $\underline{\mathcal{C}}$ *is nowhere dense.*
2. $\underline{\mathcal{C}}$ *is superflat.*
3. $\mathcal{C}$ *is stable.*
4. $\underline{\mathcal{C}}$ *is stable.*
5. FO *has bounded* VC *dimension on* $\mathcal{C}$.
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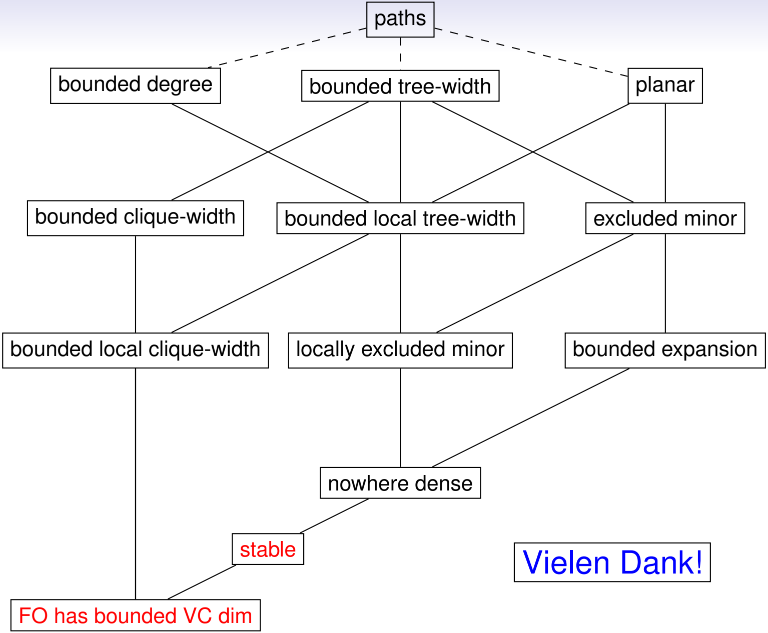
Open Problems

- Is there a simple structural characterisation for general graph classes on which FO (MSO) has bounded VC dimension?
- What about the VC dimension of other logics?
- Is first order model checking in FPT on nowhere dense graph classes?
- Explore connections between infinite model theory and algorithmic graph theory

Main sources:

[1] H. Adler, I. Adler, *Nowhere dense graph classes, stability, and the independence property*, arxiv 2010. (New version submitted)
[2] M. Grohe, Gy. Turán, *Learnability and definability in trees and similar structures*, Theory Comput. Syst. 2004.
[3] J. Nešetřil, P. Ossona de Mendez, *On nowhere dense graphs*, submitted.
[4] K.-P. Podewski, M. Ziegler, *Stable graphs*, Fund. Math. 1978.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Vielen Dank!

Why 'nowhere dense'?

Let $\mathcal{C}$ be a graph class.
$\mathcal{C}\nabla r :=$ class of all topological r-minors of graphs in $\mathcal{C}$.

Then: $\mathcal{C}\nabla 0 = \{$ all subgraphs of graphs in $\mathcal{C}\}$.

Theorem (Nešetřil, de Mendez, 2008)
$\mathcal{C}$ a class of finite graphs. Then

$$\lim_{r\to\infty} \limsup_{\substack{H\in\mathcal{C}\nabla r\\ |V(H)|\to\infty}} \frac{\log|E(H)|}{\log|V(H)|} \quad \in \quad \{0,1,2\}.$$

Moreover, the quadratic case (right-hand side 2) is equivalent to: for some r there is no finite upper bound on the sizes of cliques that occur as topological r-minors.

Original definition: **nowhere dense**:= right hand side ≤ 1.

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